Second-order approximations to pricing and hedging in presence of jumps and stochastic volatility

Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität zu Kiel

vorgelegt von

Stephan Denkl

Kiel, 2013

Erster Gutachter:	Prof. Dr. Jan Kallsen
Zweiter Gutachter:	Prof. Dr. Albrecht Irle
Tag der mündlichen Prüfung:	14. Mai 2013
Zum Druck genehmigt:	14. Mai 2013

gez. Prof. Dr. Wolfgang J. Duschl, Dekan

Zusammenfassung

In dieser Arbeit betrachten wir zwei grundlegende Probleme der Finanzmathematik, nämlich die Bewertung und die Absicherung (Hedging) von Derivaten mit europäischer Auszahlungsstruktur. Wir führen unsere Analyse in Modellen durch, die das dem Derivat zugrunde liegende Wertpapier durch einen Prozess mit Sprüngen und stochastischer Volatilität abbilden. Dabei interessieren wir uns nicht für exakte Lösungen der genannten Probleme, sondern für sinnvolle Näherungslösungen mit dem Hauptziel, einen besseren Einblick in die Struktur der jeweiligen Frage zu erhalten.

Genauer betrachten wir Hedgingprobleme in geometrischen Lévy-Modellen, das heißt in Modellen, in denen der logarithmische Wertpapierkurs einem Prozess mit unabhängigen und stationären Zuwächsen folgt. In solchen Modellen existieren typischerweise keine perfekten Absicherungsstrategien. Das Restrisiko einer selbstfinanzierenden Hedgingstrategie bewerten wir durch den *mean squared hedging error*, das heißt durch das zweite Moment der Differenz von Derivatauszahlung und Endwert der Absicherungsstrategie.

Die Frage der Derivatbewertung studieren wir in einer großen Modellklasse, die geometrische Lévy-Modelle, aber auch diverse stochastische Volatilitätsmodelle aus der Literatur umfasst. Dabei betrachten wir mit Arbitragefreiheit verträgliche Preise, das heißt solche, die risikolose Gewinne nicht zulassen.

Für verschiedene Hedgingstrategien, für deren Hedgefehler sowie für Derivatpreise existieren für den von uns betrachteten Rahmen semi-explizite Darstellungen in der Literatur, die sich für viele parametrische Modelle numerisch effizient auswerten lassen. Allerdings erlauben diese Darstellungen wenig Einsicht zum Beispiel in die für die jeweilige Größe entscheidenden Einflussfaktoren. Wir entwickeln in dieser Hinsicht besser interpretierbare Näherungslösungen, die wir durch Perturbationstechniken gewinnen. Dazu fassen wir das komplexe Modell mit Sprüngen und stochastischer Volatilität als Störung eines einfachen Black-Scholes-Modells auf und berechnen Korrekturterme zweiter Ordnung. Ein wesentlicher Unterschied zu klassischen Perturbationsansätzen besteht darin, dass in unserem Fall kein dem Problem immanenter univariater Parameter existiert, der die Störung quantifiziert. Wir entwickeln deshalb zunächst einen allgemeinen Rahmen für Perturbationstechniken in dieser Situation und wenden diesen dann in den betrachteten Modellen an.

Die so gewonnenen Näherungslösungen setzen sich aus wenigen Momenten von Komponenten des Wertpapierprozesses sowie aus Sensitivitäten (*greeks*) des Black-Scholes-Preises des betrachteten Derivats zusammen. Die Näherungen hängen insbesondere nicht von der Feinstruktur des betrachteten Modells ab und sind in diesem Sinne robust. In ausführlichen numerischen Experimenten zeigen wir, dass unsere Approximationen in verschiedenen parametrischen Modellen aus der Literatur zufriedenstellende Ergebnisse liefern.

Abstract

This thesis deals with two basic problems of Mathematical Finance, namely the pricing and hedging of European-style derivatives. We analyze these questions in models that describe the asset underlying the derivative by a process with jumps and stochastic volatility. However, we are not interested in exact solutions to the mentioned problems but in reasonable approximations that allow for a better insight into the structure of the respective question.

More precisely, we consider hedging problems in geometric Lévy models, i.e., in models where the logarithmic price process of the underlying follows a process with independent and stationary increments. In this kind of models, there typically exist no perfect hedging strategies. We quantify the remaining risk of a self-financing trading strategy by its *mean squared hedging error*, i.e., the second moment of the difference between the payoff of the derivative and the terminal wealth of the hedging portfolio.

We study the question of derivative pricing in a comprehensive model class that encompasses geometric Lévy models and several stochastic volatility models from the literature. In doing so, we consider prices that are compatible with the absence of arbitrage, i.e., prices that do not allow for riskless gains.

For several hedging strategies, for their hedging errors, as well as for derivative prices in the described framework, the literature provides semi-explicit representations that can be efficiently evaluated numerically for many parametric models. However, these representations admit little understanding, e.g., of the determining factors of the respective quantity. We develop approximate solutions that provide more insight in this respect. To this end, we interpret the complex model with jumps and stochastic volatility as a perturbed Black-Scholes model, and we compute correction terms of second order. Our approach differs from traditional perturbation techniques in the sense that in our case, there is no univariate problem-inherent parameter that quantifies the amount of perturbation. Therefore, we develop a general framework for perturbation approaches in this situation, and we apply this approach in the models under consideration.

The approximate solutions obtained in this way consist of few moments of components of the asset price process as well as of sensitivities (*greeks*) of the Black-Scholes derivative price. In particular, the formulas do not depend on the fine structure of the considered model and are robust in this sense. We show in detailed numerical experiments that our approximations yield satisfactory results in several parametric models from the literature.

Contents

List of Figures

List of Tables

1. Introduction

1.1. Approximate approaches in Mathematical Finance

> "Perturbation analysis is a very powerful tool of applied mathematics. It is used to great effect in areas such as fluid dynamics (Hinch 1991), because it reveals the salient features of the problem while remaining a good approximation to the full but more complicated model. As yet, the technique has, to our knowledge, rarely been used in finance." ([WW97, Section 1])

This quotation stems from the contribution [WW97] by Whalley & Wilmott from 1997, where the authors consider a Black-Scholes model with proportional transaction costs and deal with

Utility indifference pricing and hedging. The basic idea of the utility indifference approach is to determine the initial price of an option such that optimal trading with and without selling the option amounts to the same expected utility of terminal wealth. The corresponding optimal trading strategy with the obligation to deliver the option payoff at the end can then be used as hedge. The solution to this optimization problem is given in terms of a free boundary value problem. It can only be solved numerically in a rather time consuming way. Therefore, the authors of [WW97] employ a perturbation approach: they interpret the complex problem with transaction costs as a perturbation of a simpler problem – utility indifference pricing in the Black-Scholes model without transaction costs. In order to account for this perturbation, correction terms up to the third order with respect to the size of transaction costs are derived. These can be computed numerically in a much quicker way than the original free boundary value problem. Moreover, the resulting approximate solution "reveals the salient features of the problem". [BS98] is a further reference for an early study of expansions of utility indifference prices and hedges with respect to small proportional transaction costs.

Since 1997, approximate approaches to problems in Mathematical Finance – mostly by use of perturbation techniques – have attained increasing attention in the literature. Let us give some further examples from different fields of Mathematical Finance.

Portfolio optimization under transaction costs. When considering optimal portfolio choice and consumption under transaction costs (without an option pricing or hedging problem), the solution cannot be obtained explicitly even in simpler models. It is typically stated in terms of quasi-variational inequalities resp. free boundary value problems, cf. [EH88, DN90, SS94, MP95]. To shed more light on the structure of the problem, expansions with respect to the size of transaction costs were considered, e.g., by [Kor98, JS04, AW95].

Utility indifference pricing and hedging without transaction costs. Also in the situation without transaction costs, utility indifference prices and hedges as explained above are typically hard to obtain even for simpler models and utility functions. Therefore, first-order approximations with respect to the number of sold claims were derived as a way out, cf. [MS05, KS06, KS07].

Hedging errors. In market models exhibiting incompleteness due to jumps, stochastic volatility, or frictions in trading, an important issue is the study of the hedging error $H - V_T(\varphi)$ for an option with payoff H at maturity $T > 0$, hedged by a trading strategy φ with terminal value $V_T(\varphi)$. Besides the whole distribution of this hedging error, an important quantification of the hedging risk is the *mean squared hedging error*, given by $\mathrm{E}\left((H - V_T(\varphi))^2\right)$. As for hedging errors, the literature focuses on the effect of discrete-time hedging. An early contribution is [Tof96], which studies the mean squared hedging error of the discretely implemented delta strategy in the Black-Scholes model, deriving a first-order approximation to the error with respect to the hedging interval. [Zha99] generalizes this result to Markovian diffusion models, for which [BKL00] consider also convergence in law of the renormalized hedging error as a random variable. Extensions to irregular payoffs and more general diffusion models are to be found in [HM05, GT01, Tem03]. [Gei02, GG06] study how the convergence rate of the squared discretization error can be improved by using non-equidistant hedging times. For underlying models with jumps, [TV09a] examine the convergence rate of the discretization error of general strategies in Lévy-Itō models. [BT11] study the expected squared discretization error of the variance-optimal and the delta hedge in geometric Lévy models.

No-arbitrage option pricing. The literature on approximations to no-arbitrage option prices – i.e., prices that do not allow for riskless gains – is quite vast. E.g., [WDAN05] expand prices in the Black-Scholes model with respect to volatility. [FPS00, FPSS03, Alò06, Fuk11b, Fuk11a, KY05] consider expansions of option prices when the rate of mean reversion in a bivariate stochastic volatility diffusion model is fast, [Lew00, BGM10b] derive an expansion with respect to volatility of volatility, and [AS09] provide a power series expansion of the price with respect to correlation. [HW99, BGM09, BGM10a, PPR13] consider local volatility models and derive approximate pricing formulas essentially by a Taylor expansion of the local volatility function. In our Section 5.6, we consider some of the above and further contributions in more detail.

1.2. Why considering approximations?

In light of the examples from the preceding section, we see three main reasons why to take approximate approaches in Mathematical Finance.

1. **Tractability.** If an interesting problem is not tractable in its full complexity, approximate solutions are a way out.

2. **Computational speed and simplicity.** Even if possible, the numerical computation of exact solutions can be too slow, e.g., for calibration or risk management problems. In contrast, approximations are often analytic expressions or at least quicker to evaluate. Moreover, practitioners typically prefer simple and easy-to-implement approximations over complicated exact solutions.

3. **Structural insight and model robustness.** Approximations obtained by perturbation typically provide more insight into the structure of a problem and its main driving factors. Strongly related to this issue is the aspect of model robustness: the driving factors entering the approximation often do not depend on the fine but only on the coarse structure of the model.

1.3. Contribution of this work

In this thesis, we contribute to the stream of approximations in Mathematical Finance in two fields: hedging errors and no-arbitrage option pricing. More precisely, we provide approximations to several hedging strategies for a European option as well as to the resulting mean squared hedging errors when the underlying S follows a geometric Lévy process. I.e.,

$$S_t = S_0 e^{L_t}, \quad t \in \mathbb{R}_+,$$

for a process L with stationary and independent increments. We treat the question of no-arbitrage pricing of a European option in geometric Lévy models with stochastic volatility. Here, the underlying S is essentially given by

$$S_t = S_0 e^{L_{\int_0^t y_s \, \mathrm{d}s}}, \quad t \in \mathbb{R}_+,$$

where L is as well a process with stationary and independent increments, and y is a positive stochastic process representing stochastic trading intensity, i.e., stochastic volatility.

We obtain our approximations to hedging errors and option prices by considering the respective underlying process S as a perturbed Black-Scholes process. Since we do not assume that this perturbation is controlled by a real-valued parameter – like the size of transaction costs in our introductory example – we first develop a general framework for perturbation approaches in this situation. This framework is then applied to the two problems under consideration.

Our approximations are expressed in terms of few moments of L_1 and $\int_0^t y_s \, \mathrm{d}s$ as well as sensitivities of the Black-Scholes price of the option.

As for hedging errors, our approach and results are – to the best of our knowledge – completely new in the literature. Our approximation to option prices contains some other approximations from the literature as special cases. However, the generality of our setup, in particular the incorporation of jumps, is – to the best of our knowledge – to be found nowhere else. Moreover, our results are interesting in view of all three aspects mentioned in Section 1.2:

1. For hedging errors and option prices in our framework, semi-explicit representations are available that depend on the characteristic function of $\log(S_T)$. If it is known, errors and prices can be computed numerically. However, if the characteristic function is not available, one may still use our formula if one disposes of few moments of the underlying process.

2. If the characteristic function of $\log(S_T)$ can be evaluated, the numerical computation of hedging errors requires the quadrature of a double integral over $\mathbb{R}^2$, and for option prices a single integral over $\mathbb{R}$. For simpler options like calls and puts, the evaluation of our approximation to hedging errors needs only the numerical integration of an explicit function over a compact interval. Our approximation to option prices is even an analytic expression for many relevant models. Hence, our formulas may significantly speed up the numerical computations.

3. Since the model enters our approximations only via few moments, the formulas do not depend on the fine structure of the model and are robust in this sense. As for structural insight, we may, e.g., infer from our approximation to hedging errors that the risk of the Black-Scholes delta hedge implemented continuously in the presence of jumps approximately amounts to the same risk as if the Black-Scholes delta is employed discretely in a Black-Scholes market at time steps
$$\Delta t = \frac{1}{2}\left(\text{ExcessKurtosis}(L_1) - \text{Skewness}(L_1)^2\right).$$
From our approximation to option prices, e.g., we see that the smile of implied volatility in geometric Lévy models is approximately controlled by ExcessKurtosis(L_1), while the skew is determined by Skewness(L_1).

1.4. Organization of the thesis

This thesis is organized as follows. In Chapter 2, we present our general perturbation approach when there is no natural parameter – like the size of transaction costs – that controls the amount of perturbation. Chapter 3 reviews Laplace transform techniques, which play a key role for our analysis in the subsequent chapters. We apply our general perturbation approach in Chapter 4 to derive approximations to several hedging strategies and their mean squared hedging errors. We asses the quality of our approximations in different parametric models from the literature. In Chapter 5, we develop approximations to no-arbitrage option prices and implied volatilities, using again our framework from Chapter 2. We demonstrate that our setup encompasses a large class of important parametric models and perform detailed numerical tests. Moreover, we review several alternative approximations from the literature.

Throughout, lengthy or technical proofs are delegated to the end of the respective chapter for the ease of exposition.

Appendix A contains some technical lemmas from stochastic calculus that are used in the proofs of Chapter 5. For use in our numerical tests, we derive explicit expressions for moments of the integrated square root and the integrated squared Gaussian Ornstein-Uhlenbeck process in Appendix B. Appendix C recaps sufficient conditions when differentiation and integration can be interchanged, which we need in several places in our proofs. Explicit representations for sensitivities of call and put prices in the Black-Scholes model are derived in Appendix D. Finally, Appendix E contains important facts about differential semimartingale characteristics and exponential compensators for use in Chapter 5.

1.5. Notation and mathematical background

By $\mathbb{R}$ and $\mathbb{C}$ we denote the real and complex numbers. $\mathbb{R}_+$ and $\mathbb{R}_-$ are the non-negative and the non-positive real numbers including 0. We denote the natural numbers including 0 by $\mathbb{N}$, and $\mathbb{N}_{\geq k} := \mathbb{N}\backslash\{0,\ldots,k-1\}$ for $k \in \{1,2,\ldots\}$. $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ denote the real and imaginary part of a complex vector $z \in \mathbb{C}^d$. By I we denote the identity process, i.e., $I_t = t$ for $t \in \mathbb{R}_+$. For real intervals A and B, we denote by $C^\infty(A,B)$ the set of infinitely often differentiable functions from A to B. If A is closed at its left or right boundary, the derivatives are to be understood as one-sided.

All unexplained notation and terminology concerning stochastic processes is as in [JS03]. For background on Lévy processes, we refer the reader also to the monograph [Sat99].

2. Perturbation approach employed in this thesis

In this chapter, we describe in general terms the perturbation approach that we employ to derive approximations to hedging and pricing problems in Chapters 4 and 5.

2.1. Perturbation in presence of a natural small parameter

The essential idea of perturbation theory is to interpret a complex problem as a – in some sense – small deviation from a simpler situation, which can be dealt with more easily. An approximate solution to the complex problem is then obtained by the solution to the simple problem plus corrections that approximately account for the perturbation. The amount of the deviation is usually quantified by a parameter inherent in the complex problem.

This approach is more and more employed in Mathematical Finance, where it is quite common to consider a complex situation as perturbation of a simple Black-Scholes environment. The deviation from Black-Scholes is typically quantified by a small univariate parameter ε that can be interpreted directly in the context of the problem. Let us mention only three examples:

1. Option pricing and hedging in the Black-Scholes model with proportional transaction costs of size $\varepsilon > 0$ (cf. [WW97, BS98]),
2. Hedging in the Black-Scholes model at discrete points in time with distance $\varepsilon > 0$ (cf. [Tof96, Zha99, BKL00, GT01]),
3. Option pricing in stochastic volatility diffusion models, where the volatility has mean reversion speed $1/\varepsilon > 0$ (cf. [FPS00, FPSS03, Fuk11b, Fuk11a]).

Suppose that we are interested in a certain quantity $q(\varepsilon)$ of the perturbed Black-Scholes model with small parameter ε. Referring to the examples above, such quantity could be the indifference price and hedge under transaction costs of size $\varepsilon > 0$, the mean squared hedging error of discrete delta hedging at time steps $\varepsilon > 0$, or the option price under stochastic volatility with mean reversion speed $1/\varepsilon > 0$. If ε is sufficiently small, we will typically expect the first- or at least the second-order expansion

$$q(\varepsilon) \approx q(0) + q'(0)\varepsilon \tag{2.1}$$

resp.

$$q(\varepsilon) \approx q(0) + q'(0)\varepsilon + \frac{1}{2}q''(0)\varepsilon^2 \tag{2.2}$$

to provide a reasonable approximation. Here, $q(0)$ is the respective quantity in the Black-Scholes model itself, which is typically known explicitly. Under sufficient regularity, the approximations to $q(\varepsilon)$ are good if ε is small.

2.2. Perturbation with respect to an artificial parameter

In this work, we are interested in approximations to real-valued quantities relative to a stock price process S. Such quantity is, e.g., the mean squared hedging error of a certain hedging strategy (cf. Chapter 4) or the no-arbitrage price of the option with payoff $f(S_T)$ for some maturity $T > 0$ and payoff function f (cf. Chapter 5). We consider S to be the exponential of a Lévy process or the exponential of a time-changed Lévy process. In order to obtain reasonable approximations to the quantity of interest $Q \in \mathbb{R}$, we also interpret S as perturbation of a Black-Scholes price process, and we compute approximate corrections of the Black-Scholes value of the quantity to account for the perturbation.

However, our situation is quite different from the one outlined in Section 2.1. We do not assume that S belongs to a specific parametric class of, e.g., Lévy processes, and hence there is no natural small parameter that captures the deviation of the stock price process S from geometric Brownian motion. Then, the approach corresponding to (2.1) resp. (2.2) does not seem to make sense. As a way out, we introduce an artificial parameter $\lambda \in [0,1]$, where

- $\lambda = 1$ corresponds to the original stock price process S of interest,
- $\lambda = 0$ corresponds to a Black-Scholes model whose first two moments fit to those of the original model of interest, and
- $\lambda \in (0,1)$ corresponds to an interpolation between the two cases above, which will be specified in the concrete applications in Chapters 4 and 5.

Put differently, we connect the stock price process of interest S with the Black-Scholes setup via a curve in the space of stochastic processes, parametrized by $\lambda \in [0,1]$. This yields a whole family of stock price processes S^λ, $\lambda \in [0,1]$. Let now $q(\lambda)$ denote the quantity of interest in the model corresponding to parameter value $\lambda \in [0,1]$, e.g., the price of the option with payoff $f(S_T^\lambda)$, i.e., $q(\lambda) = \mathrm{E}\left(f(S_T^\lambda)\right)$. We then suggest to use the second-order expansion corresponding to (2.2) as an approximation to Q in our stock price model of interest, i.e., for the parameter $\lambda = 1$, which means

$$Q = q(1) \approx q(0) + q'(0) + \frac{1}{2}q''(0). \tag{2.3}$$

Let us summarize this idea in the following

Principle 2.2.1. Let Q denote a real-valued quantity relative to the stock price process S of interest, and let S^λ, $\lambda \in [0,1]$, be a family of stock price processes such that $S^1 = S$ and such that S^0 is geometric Brownian motion. Denote by $q(\lambda)$, $\lambda \in [0,1]$, the corresponding quantity with respect to S^λ. If $\lambda \mapsto q(\lambda)$ is twice continuously differentiable, we call

$$\mathfrak{A}(Q) := \mathfrak{A}_0(Q) + \mathfrak{A}_1(Q) + \frac{1}{2}\mathfrak{A}_2(Q) \tag{2.4}$$

with

$$\mathfrak{A}_0(Q) := q(0), \quad \mathfrak{A}_1(Q) := q'(0), \quad \mathfrak{A}_2(Q) := q''(0)$$

second-order approximation to Q (relative to the curve S^λ).

2.3. Discussion of our approach

Let us first mention that our perturbation with respect to an artificial parameter contains the approach in presence of a natural parameter in the following sense: for the given natural parameter ε, consider the family of problems with parameter $\lambda\varepsilon$, $\lambda \in [0,1]$, instead of ε. Relative to the artificial parameter $\lambda \in [0,1]$, the quantity of interest is given by $q(\lambda\varepsilon)$. In this case, our approximation (2.3) reads

$$q(1\cdot\varepsilon) \approx \mathfrak{A}(Q) = q(0\cdot\varepsilon) + q'(0\cdot\varepsilon)\varepsilon + \frac{1}{2}q''(0\cdot\varepsilon)\varepsilon^2,$$

which is (2.2).

Moreover, we stress that both the specific form and the quality of the approximation (2.4) clearly depend on the choice of the curve that connects geometric Brownian motion with the stock price process S of interest. Hence, our approach is to be understood as a general framework to obtain approximations in situations without a natural parameter. Besides the computation of $q'(0)$ and $q''(0)$, the choice of the curve is to be considered as part of the problem. However, such choice should satisfy two qualitative properties in order to yield reasonable results:

1. For the quantity $q(\lambda)$ of interest relative to S^λ, the derivatives $q'(0)$, $q''(0)$ of course need to exist and should be computable as explicitly as possible.

2. The error of approximation (2.4) must be reasonably small in practically relevant cases.

Since explicit and reasonably tight error bounds are typically hard to come by, one will have to test the second property in numerical experiments.

Finally, we mention that one can, of course, consider also approximations of higher order than two relative to the artificial (and the natural) parameter, provided the corresponding derivatives exist. As put forward in Principle 2.2.1, we will always work with second-order approximations in this thesis since these turned out to have a reasonable trade-off between accuracy and computability in our applications.

2.4. Related approaches in the literature

The series of papers [BGM09, BGM10a, BGM10b] is the only reference we are aware of where an approach similar to ours is taken. There, the authors consider the problem of computing approximations to option prices in general local volatility models and the Heston model with deterministic but time dependent parameter functions. As in our case, there is no natural univariate parameter describing the deviation from Black-Scholes. The authors therefore introduce an artificial parameter that parametrizes a curve connecting the original model with a simpler one, and they expand the option price relative to this parameter. For a more detailed discussion, we refer to Section 5.6.5 below in this thesis.

3. Laplace transform approach

In this chapter, we review Fourier-Laplace techniques since these are essential to our considerations in Chapters 4 and 5. Moreover, we derive several related results.

3.1. Basic principle

Let us consider a market with two assets, a bank account constantly equal to 1 and a non dividend paying stock whose price process is given by a positive stochastic process S. Moreover, we consider a European option on S with payoff $f(S_T)$ at maturity $T > 0$, where $f : \mathbb{R}_+ \to \mathbb{R}$. By the fundamental theorem of asset pricing [DS94], the only reasonable initial prices of the option – in the sense that they do not admit arbitrage – are given by

$$V_0 = \mathrm{E}_P(f(S_T)), \tag{3.1}$$

where P is an equivalent martingale measure of S. In the following, we assume to be given the dynamics of S relative to such an equivalent martingale measure P, and we are interested in the computation of the corresponding price V_0.

To fix ideas, S could be the exponential of a Lévy process relative to P, and f could be the payoff function of a European call option with strike $K > 0$, i.e., $f(s) = (s-K)^+$. If the density of S_T is known, the expectation in (3.1) can be computed at least numerically. However, for many stock price models of practical importance, one does not dispose of the density of S_T or $\log(S_T)$.

A way out is the Laplace transform approach, of which the key idea is to assume that the payoff function f allows for the representation

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^z \, p(z) \, \mathrm{d}z, \quad s \in \mathbb{R}_+, \tag{3.2}$$

for suitable $R \in \mathbb{R}$ and a weight function $p : (R+i\mathbb{R}) \to \mathbb{C}$ such that $\int_{-\infty}^{\infty} |p(R+ix)| \, \mathrm{d}x < \infty$. Intuitively speaking, (3.2) allows to represent the complicated payoff $f(S_T)$ as a generalized linear combination of simple payoffs of the form S_T^z, $z \in R+i\mathbb{R}$. To solve the complicated pricing problem, it is sufficient to consider the pricing problems for the simpler payoffs since

$$V_0 = \mathrm{E}(f(S_T)) = \mathrm{E}\left(\int_{R-i\infty}^{R+i\infty} S_T^z \, p(z) \, \mathrm{d}z\right) = \int_{R-i\infty}^{R+i\infty} \mathrm{E}(S_T^z) \, p(z) \, \mathrm{d}z,$$

assuming for the moment that interchange of expectation and integration with respect to z is possible. Observe that

$$\mathrm{E}\left(S_T^z\right) = \mathrm{E}\left(e^{z\log(S_T)}\right) = \varphi_{\log(S_T)}(z),$$

where $\varphi_{\log(S_T)}$ denotes the extended characteristic function of $\log(S_T)$. To sum up, these considerations yield

$$V_0 = \int_{R-i\infty}^{R+i\infty} \varphi_{\log(S_T)}(z) p(z)\,\mathrm{d}z. \tag{3.3}$$

In contrast to the density, the (extended) characteristic function of $\log(S_T)$ is known in closed form for many stock price models of importance, e.g., for geometric Lévy models, it is given directly via the Lévy-Khintchine triplet. For models belonging to the class of affine processes, it can be obtained by the solution of generalized Ricatti differential equations [DFS03]. The integral in (3.3) can then be evaluated efficiently and accurately via numerical integration. Moreover, (3.3) represents the option price in a way that disentangles the model and the payoff function.

Let us summarize our so far formal considerations on the integral representation of option prices in the following theorem. We consider not only the initial price but the whole price process.

Theorem 3.1.1 (Integral representation of option prices)**.** *Let S be a positive adapted stochastic process on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$, and let $f : \mathbb{R}_+ \to \mathbb{R}$. Assume that there are $R \in \mathbb{R}$ and $p : (R + i\mathbb{R}) \to \mathbb{C}$ with $\int_{-\infty}^{\infty} |p(R+ix)|\,\mathrm{d}x < \infty$ such that*

1. *f admits the representation*

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^z\, p(z)\,\mathrm{d}z, \quad s \in \mathbb{R}_+,$$

2. *and $\mathrm{E}(S_T^R) = \mathrm{E}(e^{R\log(S_T)}) < \infty$.*

Then, $f(S_T) \in L^1(P)$, and

$$\mathrm{E}(f(S_T)|\,\mathscr{F}_t) = \int_{R-i\infty}^{R+i\infty} \varphi_{\log(S_T)|\mathscr{F}_t}(z)\, p(z)\,\mathrm{d}z, \tag{3.4}$$

where $\varphi_{\log(S_T)|\mathscr{F}_t}$ denotes the extended conditional characteristic function of $\log(S_T)$, i.e.,

$$\varphi_{\log(S_T)|\mathscr{F}_t}(y) = \mathrm{E}\left(e^{y\log(S_T)}\,\middle|\,\mathscr{F}_t\right)$$

for $y \in \mathbb{C}$ such that $\mathrm{E}\left(e^{\mathrm{Re}(y)\log(S_T)}\right) < \infty$.

PROOF. Cf. Section 3.6. □

The approach outlined above is not limited to no-arbitrage pricing problems, but it can be fruitfully applied whenever the problem at hand is linear in the option. In order to illustrate this, consider the example of *variance-optimal hedging*. This framework goes back to [FS86] and aims at finding a reasonable self-financing hedging strategy for a payoff $f(S_T)$ that is not replicable. More precisely, the idea is to find the solution of the minimization problem

$$\min_{c,\psi} \mathrm{E}\left(\left(f(S_T) - c - \int_0^T \psi_t \,\mathrm{d}S_t\right)^2\right) \tag{3.5}$$

for all initial capitals $c \in \mathbb{R}$ and in some sense admissible trading strategies ψ in the stock. For more detailed introductions to the topic, we refer to [Pha00, Sch01]. From the analytic point of view, the optimizer (c^*, ψ^*) is the L^2-projection of the payoff random variable $f(S_T)$ on the subspace of random variables $\{c + \int_0^T \psi_t \,\mathrm{d}S_t : c \in \mathbb{R}, \psi \text{ trading strategy}\}$, i.e., all terminal wealths of admissible self-financing portfolios. If f admits Representation (3.2), the linearity of projections suggests that the solution to the complicated problem (3.5) is given by

$$c^* = \int_{R-i\infty}^{R+i\infty} c^*(z)\, p(z)\,\mathrm{d}z, \quad \psi^* = \int_{R-i\infty}^{R+i\infty} \psi^*(z)\, p(z)\,\mathrm{d}z, \tag{3.6}$$

where $(c^*(z), \psi^*(z))$ denotes the solution to the variance-optimal hedging problem for the simpler payoff S_T^z, which is often more feasible. This idea is made precise in [HKK06] for geometric Lévy models and in [KP10, KV09] for affine stochastic volatility models.

We have seen that the outlined approach allows for the exact computation of e.g. option prices up to numerical integration. Even though in this work we are not interested in the exact computation of such quantities but in approximations to them, representations like (3.3) and (3.6) turn out to be a fruitful starting point for our considerations in the subsequent chapters.

3.2. Laplace transform and inversion formula

In the last section, we have seen that Representation (3.2) of the payoff function f is very useful to solve pricing and hedging problems. Such a representation is typically obtained by computing the Laplace or Fourier transform of (a modification of) f and by applying a suitable inversion formula.

Definition 3.2.1. [Laplace transform] Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable function. The (bilateral) Laplace transform $\mathscr{L}[g;z]$ of g at the point $z \in \mathbb{C}$ is given by

$$\mathscr{L}[g;z] := \int_{-\infty}^{\infty} g(x) e^{-zx}\,\mathrm{d}x,$$

supposed that the integral exists in the sense that the Lebesgue integral $\int_{-\infty}^{\infty} |g(x)e^{-zx}|\,\mathrm{d}x$ is finite.

Theorem 3.2.2 (Bromwich inversion formula)**.** *Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable function, and suppose that $\mathscr{L}[g;R]$ exists for some $R \in \mathbb{R}$.*

1. *If $v \mapsto \mathscr{L}[g;R+iv]$ is Lebesgue-integrable on $\mathbb{R}$, then g is continuous, and*

$$g(x) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} \mathscr{L}[g;z]\, e^{zx}\,\mathrm{d}z, \quad x \in \mathbb{R}.$$

2. *If g is of finite variation at $x \in \mathbb{R}$, then*

$$\lim_{\varepsilon \to 0} \frac{1}{2}\left(g(x+\varepsilon)+g(x-\varepsilon)\right) = \lim_{c\to\infty} \int_{R-ic}^{R+ic} \mathscr{L}[g;z]\, e^{zx}\, \mathrm{d}z.$$

PROOF OF THEOREM 3.2.2. The first assertion follows from [Rud87, Theorem 9.11], where the corresponding statement is formulated for the Fourier transform (cf. also Remark 3.2.4 below). For the second assertion, cf. [Doe50, Satz 4.4.1]. □

Representation (3.2) for a payoff function f can then be obtained from Theorem 3.2.2(1): if there exists $R \in \mathbb{R}$ such that $v \mapsto \mathscr{L}[f \circ \exp; R+iv]$ is integrable, then

$$f(e^x) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} \mathscr{L}[f\circ\exp;z]\, e^{zx}\,\mathrm{d}z, \quad x \in \mathbb{R}.$$

Substituting $s = e^x$ yields

$$f(s) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} \mathscr{L}[f\circ\exp;z]\, s^{z}\,\mathrm{d}z, \quad s \in \mathbb{R}_+,$$

and Representation (3.2) is given by setting $p(z) := \frac{1}{2\pi i}\mathscr{L}[f\circ\exp;z]$.

Example 3.2.3. [Integral representation of European call] For the payoff function $f(s) = (s-K)^+$ of a European call option with strike $K > 0$, direct computation shows that

$$\mathscr{L}[f\circ\exp;z] = \frac{K^{1-z}}{z(z-1)}$$

for $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 1$. Hence, $v \mapsto \mathscr{L}[f\circ\exp;R+iv]$ is integrable for all $R > 1$, and Theorem 3.2.2(1) yields

$$f(s) = (s-K)^+ = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)}\,\mathrm{d}z.$$

The payoff function of the European put $s \mapsto (K-s)^+$ can be represented by the same formula but with $R < 0$.

Integral representations for most European payoff options of practical importance can be obtained in this manner. We refer to [Rai00, Chapter 3] and [HKK06, Section 4] for further examples.

It may occur that $v \mapsto \mathscr{L}[f\circ\exp;R+iv]$ is finite but not integrable, and hence Theorem 3.2.2(1) cannot be applied. E.g., this is the case for the payoff function of the digital option $s \mapsto 1_{[K,\infty)}(s)$ with strike $K > 0$, which coincides almost everywhere with $f(s) = \frac{1}{2}1_{\{K\}}(s) + 1_{(K,\infty)}(s)$. Considering the latter function is sufficient if S_T has no atoms, which is typically the case. One directly verifies that

$$\mathscr{L}[f\circ\exp;z] = \frac{K^{-z}}{z}$$

for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Theorem 3.2.2(2) then yields

$$f(s) = \lim_{c\to\infty}\frac{1}{2\pi i}\int_{R-ic}^{R+ic} s^z \frac{K^{-z}}{z}\,\mathrm{d}z, \quad s \in \mathbb{R}_+.$$

Integral representations of, e.g., option prices typically carry over correspondingly. However, in the remainder of this work, we will always assume that f allows for Representation (3.2).

Remark 3.2.4. Denoting by $\mathscr{F}[g;y] := \int_{-\infty}^{\infty} g(x)e^{-ixy}\,\mathrm{d}x$ the Fourier transform at $y \in \mathbb{R}$ of an integrable function $g : \mathbb{R} \to \mathbb{C}$, we see that

$$\mathscr{L}[g;u+iv] = \mathscr{F}\left[x \mapsto e^{-ux}g(x);v\right].$$

Hence, all properties of the Laplace transform can be expressed in terms of the Fourier transform and vice versa. In the Mathematical Finance literature on transform techniques, it is more common to work with the Fourier transform of the (suitably dampened) payoff function. However, we prefer to present the theory in terms of the Laplace transform.

3.3. Smooth payoff function

In Chapter 4 on approximate hedging, we work with payoff functions allowing for Representation (3.2). To perform our analysis, we need the weighting function p not only to be integrable but to admit arbitrary moments. A sufficient condition on the payoff function for this to hold is provided by the following

Lemma 3.3.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be infinitely often differentiable, and assume that there exists $R \in \mathbb{R}$ such that all derivatives of $x \mapsto f(e^x)e^{-Rx}$ are Lebesgue-integrable. Then, there exists a function $p : (R+i\mathbb{R}) \to \mathbb{C}$ with $\int_{-\infty}^{\infty} |p(R+ix)|\,\mathrm{d}x < \infty$ such that f admits the representation*

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^z p(z)dz, \quad s \in \mathbb{R}_+. \tag{3.7}$$

Moreover, $x \mapsto |R+ix|^n\,|p(R+ix)|$ is integrable for all $n \in \mathbb{N}$.

PROOF. Cf. Section 3.6. □

3.4. Cash greeks in the Black-Scholes model

Integral Representation (3.4) of the option price allows to derive related representations for sensitivities, i.e., derivatives of the price with respect to underlying variables. Such sensitivities for prices in the celebrated Black-Scholes model [BS73] will be of particular interest in Chapters 4 and 5.

In this subsection, we denote by S the discounted price process of the stock in a Black-Scholes market with volatility parameter $\sigma > 0$. Relative to the unique equivalent martingale measure P, we have

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}, \quad t \in \mathbb{R}_+,$$

for the initial stock price $S_0 > 0$ and a P-standard Brownian motion W. Moreover, consider the option with discounted payoff $f(S_T)$ at maturity $T > 0$ for a payoff function $f : \mathbb{R}_+ \to \mathbb{R}$ admitting Representation (3.2) for suitable $R \in \mathbb{R}$ and $p : (R+i\mathbb{R}) \to \mathbb{C}$. The discounted price of the option at time $t \in [0,T]$ is given by $C(t,S_t)$, where the function $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$C(t,s) := \mathrm{E}\left(f(S_T)|\,S_t = s\right). \tag{3.8}$$

Later in this work, we are interested in derivatives of $C(t,s)$ with respect to s, multiplied with corresponding powers of s. Such sensitivities are referred to as *cash greeks*.

Lemma 3.4.1. *The function $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, $(t,s) \mapsto C(t,s)$, from* (3.8) *is infinitely often differentiable with respect to the second variable s for $t \in [0,T)$.*

PROOF. Cf. Section 3.6. □

Definition 3.4.2. For $n \in \mathbb{N}$, set

$$D_n(t,s) := s^n \frac{\partial^n}{\partial s^n} C(t,s), \quad t \in [0,T), s \in \mathbb{R}_+,$$

with function $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, $(t,s) \mapsto C(t,s)$, from (3.8).

From Representation (3.2) of the payoff function, we readily obtain an integral representation for cash greeks in the Black-Scholes model.

Lemma 3.4.3. *$D_n(t,s)$ in Definition 3.4.2 can be written as*

$$D_n(t,s) = \int_{R-i\infty}^{R+i\infty} \left(\prod_{i=0}^{n-1} (z-i) \right) s^z e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)} p(z) \, \mathrm{d}z$$

for any $n \in \mathbb{N}$, $t \in [0,T)$, and $s \in \mathbb{R}_+$.

PROOF. Cf. Section 3.6. □

3.5. Further references

Fourier resp. Laplace techniques for option pricing have been introduced by [CM99] and [Rai00]. In the framework of geometric Lévy models, [Lew01] interprets integral representations for call and put options as in Theorem 3.1.1 as contour integrals and shifts the line of integration across poles of the integrand, which yields new formulas due to the arising residue corrections. [Lee04] considers in a general framework several payoff function classes and derives numerous integral representations for the related prices, also by aid of the residue calculus. Moreover, he studies in detail the appropriate choice of the parameter R to minimize the error arising from numerical quadrature of the price integral. A systematic analysis of the conditions ensuring the existence of integral transform formulas for option prices in the single- and multi-dimensional case is provided by the more recent contribution [EGP10]. Textbooks discussing transform techniques in Mathematical Finance include [Sch03, CT03].

3.6. Proofs

PROOF OF THEOREM 3.1.1. First, note that 2. implies that $\varphi_{\log(S_T)|\mathscr{F}_t}(z)$ exists for all $z \in R + i\mathbb{R}$. Moreover, the mapping

$$(x, \omega) \mapsto \varphi_{\log(S_T)|\mathscr{F}_t}(R+ix)(\omega) = \mathrm{E}\left(e^{(R+ix)\log(S_T)} \middle| \mathscr{F}_t \right)(\omega)$$

on $\mathbb{R} \times \Omega$ is continuous in x for almost all $\omega \in \Omega$ by dominated convergence for the conditional expectation and $\mathscr{F}_t$-measurable in ω for all $x \in \mathbb{R}$ by definition of the conditional expectation. [Gow72, Theorem 2] yields that the mapping is even $\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}_t$-measurable. Moreover, by the triangle inequality,

$$\int_{-\infty}^{\infty} \left| \varphi_{\log(S_T)|\mathscr{F}_t}(R+ix) \right| |p(R+ix)| \,\mathrm{d}x \leq \mathrm{E}\left(e^{R\log(S_T)} \middle| \mathscr{F}_t \right) \int_{-\infty}^{\infty} |p(R+ix)| \,\mathrm{d}x, \tag{3.9}$$

and the right-hand side is finite for all $\omega \in \Omega$ since $x \mapsto |p(R+ix)|$ is integrable by assumption. Hence, the right-hand side of (3.4) is well defined and $\mathscr{F}_t$-measurable by Fubini's Theorem. Moreover, (3.9) implies that

$$\begin{aligned} \mathrm{E}\left(\left| \int_{R-i\infty}^{R+i\infty} \varphi_{\log(S_T)|\mathscr{F}_t}(z)\, p(z) \,\mathrm{d}z \right| \right) &\leq \mathrm{E}\left(\mathrm{E}\left(e^{R\log(S_T)} \middle| \mathscr{F}_t \right) \right) \int_{-\infty}^{\infty} |p(R+ix)| \,\mathrm{d}x \\ &= \mathrm{E}\left(e^{R\log(S_T)} \right) \int_{-\infty}^{\infty} |p(R+ix)| \,\mathrm{d}x, \end{aligned} \tag{3.10}$$

which is finite by assumption. Hence, the right-hand side of (3.4) is a random variable in $L^1(P)$. This is also the case for $f(S_T)$ since by 1., 2., and Fubini's Theorem

$$\begin{aligned} \mathrm{E}(|f(S_T)|) &= \mathrm{E}\left(\left| \int_{R-i\infty}^{R+i\infty} S_T^z\, p(z) \,\mathrm{d}z \right| \right) \\ &\leq \mathrm{E}\left(\int_{-\infty}^{\infty} \left| S_T^{R+ix}\, p(R+ix) \right| \,\mathrm{d}x \right) \\ &= \mathrm{E}\left(\int_{-\infty}^{\infty} S_T^R\, |p(R+ix)| \,\mathrm{d}x \right) \\ &= \mathrm{E}\left(e^{R\log(S_T)} \right) \int_{-\infty}^{\infty} |p(R+ix)| \,\mathrm{d}x \\ &< \infty. \end{aligned}$$

Let now $F \in \mathscr{F}_t$. Due to (3.10), we can apply Fubini's Theorem to see that

$$\begin{aligned} \mathrm{E}\left(1_F \int_{R-i\infty}^{R+i\infty} \varphi_{\log(S_T)|\mathscr{F}_t}(z)\, p(z) \,\mathrm{d}z \right) &= \int_{R-i\infty}^{R+i\infty} \mathrm{E}\left(1_F \varphi_{\log(S_T)|\mathscr{F}_t}(z) \right) p(z) \,\mathrm{d}z \\ &= \int_{R-i\infty}^{R+i\infty} \mathrm{E}\left(1_F e^{z\log(S_T)} \right) p(z) \,\mathrm{d}z \\ &= \mathrm{E}\left(1_F \int_{R-i\infty}^{R+i\infty} S_T^z\, p(z) \,\mathrm{d}z \right) \\ &= \mathrm{E}(1_F f(S_T)). \end{aligned}$$

By definition of the conditional expectation, this shows (3.4) and completes the proof. $\square$

PROOF OF LEMMA 3.3.1. To show the assertion, we use the well-known fact that the Fourier transform of a smooth function decays rapidly. Since a corresponding statement for the Laplace transform is hard to find in the literature, we provide the arguments using the Fourier transform. Set $l(x) := f(e^x)e^{-Rx}$. Iterated application of [Dei05, Theorem 3.3.1(f)] yields that $y \mapsto y^n \mathscr{F}[l;y]$ is integrable for all $n \in \mathbb{N}$ (cf. Remark 3.2.4 for the definition of the Fourier transform $\mathscr{F}[l;y]$). Setting $\widetilde{p}(R+iy) := \mathscr{F}[l;y]$, we have by the Fourier Inversion Theorem (cf. [Dei05, Theorem 3.4.4]) for all $x \in \mathbb{R}$

$$f(e^x) = e^{Rx} l(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathscr{F}[l;y] \, e^{(R+iy)x} \, \mathrm{d}y = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \widetilde{p}(z) e^{zx} \, \mathrm{d}z.$$

Setting $p(R+iy) := \frac{1}{2\pi i} \widetilde{p}(R+iy)$ and $s = e^x$ yields (3.7). Moreover, we have for all $n \in \mathbb{N}$ and all $y \in \mathbb{R}$

$$|R+iy|^n \, |p(R+iy)| \leq \frac{2^n}{2\pi} \max\{|R|^n, |y|^n\} \, |\mathscr{F}[l;y]| \, .$$

Hence, the assertion on the integrability of $x \mapsto |R+ix|^n \, |p(R+ix)|$ follows from the integrability of $y \mapsto y^n \mathscr{F}[l;y]$, which we derived in the beginning. □

PROOF OF LEMMAS 3.4.1 AND 3.4.3. Observe that $\mathrm{E}\left(e^{R \log(S_T)}\right) < \infty$ no matter how R is chosen since the normal distribution has all exponential moments. By Theorem 3.1.1 (choosing as filtration the one generated by S) and the Markov property of S, we have for $t \in [0,T]$

$$\mathrm{E}(f(S_T)|S_t) = \int_{R-i\infty}^{R+i\infty} \varphi_{\log(S_T)|S_t}(z) \, p(z) \, \mathrm{d}z.$$

The stationarity and the independence of the increments of W yield

$$\varphi_{\log(S_T)|S_t}(z) = S_t e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)}, \quad z \in \mathbb{C},$$

and hence for $t \in [0,T]$ and $s \in \mathbb{R}_+$

$$C(t,s) = \int_{R-i\infty}^{R+i\infty} s^z e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)} \, p(z) \, \mathrm{d}z.$$

For all $n \in \mathbb{N}$, $s \in \mathbb{R}_+$, and $t \in [0,T)$

$$\int_{R-i\infty}^{R+i\infty} \frac{\partial^n}{\partial s^n} \left(s^z e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)} \right) p(z) \, \mathrm{d}z = \int_{R-i\infty}^{R+i\infty} \left(\prod_{i=0}^{n-1} (z-i) \right) s^{z-n} e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)} \, p(z) \, \mathrm{d}z$$

is well-defined by Lemma 4.5.7 below. The same lemma allows to apply Corollary C.0.3, which yields that differentiation with respect to s and integration with respect to z can be interchanged. This shows Lemmas 4.3.1 and 4.5.2. □

4. Approximate quadratic hedging in geometric Lévy models

4.1. Introduction

A basic problem in Mathematical Finance is how the issuer of an option can hedge the resulting exposure by trading in the underlying. In complete markets, the risk can be offset completely by purchasing the replicating portfolio. In incomplete markets, however, additional criteria are necessary to determine reasonable hedging strategies. A popular approach studied intensively in the literature is *variance-optimal hedging*. Here, the idea is to minimize the *mean squared hedging error*, i.e., the second moment of the difference between the option's payoff and the terminal wealth of the hedging portfolio. Comprehensive overviews on the topic can be found in [Pha00, Sch01]. For more recent publications, the reader is referred to [ČernýK07] and the references therein.

As a model for stock price changes, we consider geometric Lévy processes, which have been widely studied both in the theoretical and empirical literature, cf., e.g., [EK95, Ryd97, BN95, MS90, MCC98, Rai00, CGMY02] and the monographs [Sch03, CT03]. In the context of variance-optimal hedging, [HKK06] and [Černý07] compute semi-explicit representations of the optimal strategy and the corresponding hedging error by means of Fourier/Laplace transform methods (cf. Chapter 3). In addition, [Černý07] calculates the error of the *locally optimal hedge*. The related study [DGMK$^+$13] derives formulas for the mean squared hedging error of alternative *suboptimal strategies* such as the Black-Scholes hedge, which is still prevalent in practice.

These results are exact and yield numerically tractable expressions in integral form. However, they are hard to interpret and do not allow to identify the key factors that contribute to the hedging error when deviating from the Black-Scholes model. In addition, they do not reveal how sensitively the hedging error and strategies depend on the choice of a particular parametric Lévy model.

In this chapter, we therefore strive for reasonable second-order approximations, which shed more light on the structure and dominating factors of hedging strategies and the corresponding hedging errors. It turns out that to second order, the Lévy process enters the solution only through its first four moments. Moreover, both strategies and hedging errors involve Black-Scholes sensitivities of the option. Depending on the payoff, the approximations are either in closed form or easy to implement numerically. In particular, they bypass the need to fit return data to a specific parametric Lévy model. A numerical study shown in Section 4.4 indicates that our formulas work well for a variety of Lévy models suggested in the empirical literature.

In order to derive these approximations, we employ our perturbation approach from Chapter 2. I.e., we interpret the Lévy model at hand as a perturbed Black-Scholes model, and we propose a curve in the space of Lévy processes that connects the driving Lévy process with Brownian motion. We then compute second-order corrections that account for the perturbation.

This chapter is organized as follows. In Section 4.2, we introduce our mathematical setup. In particular, we specify the relevant hedging strategies to which we employ our perturbation approach that leads to our approximate formulas. These are stated and discussed in Section 4.3. Subsequently, we illustrate our results for various parametric Lévy models. For the sake of a clear exposition, all proofs are deferred to Section 4.5. Section 4.6 concludes.

4.2. Mathematical setup

4.2.1. Market model

We consider a market consisting of two traded assets, a bond and a non dividend paying stock. The price process B of the bond is given by

$$B_t = e^{rt}, \quad t \in \mathbb{R}_+,$$

for a deterministic interest rate $r \geq 0$. In what follows, we will always work with discounted quantities, using B as numéraire. The discounted price process S of the stock is given by

$$S_t = S_0 e^{L_t}, \quad t \in \mathbb{R}_+, \tag{4.1}$$

for a deterministic initial stock price $S_0 > 0$ and a real-valued Lévy process L with $L_0 = 0$, defined on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$. The filtration is assumed to be generated by L.

In order to carry out our analysis, we impose the following assumption on the driving Lévy process L.

Assumption 4.2.1. We assume that

1. $\mathrm{E}\left(e^{2L_1}\right) < \infty$,
2. $\mathrm{E}\left(|L_1|^n\right) < \infty$ for $n \in \{1, \ldots, 5\}$, and
3. $\mathrm{Var}\left(L_1\right) > 0$.

Given that we study second moments of hedging errors (cf. Section 4.2.3 below) and their approximation in terms of moments of L, Requirements 1 and 2 are indispensable. The third requirement excludes the degenerate case that S is deterministic.

4.2.2. Option payoff function

For the rest of the chapter, we consider a fixed European contingent claim with discounted payoff $f(S_T)$ with maturity $T > 0$ and (discounted) payoff function $f : \mathbb{R}_+ \to \mathbb{R}$, which shall satisfy the following

Assumption 4.2.2. We assume that the payoff function $f : \mathbb{R}_+ \to \mathbb{R}$ of the contingent claim under consideration is in $C^\infty(\mathbb{R}_+, \mathbb{R})$ and that there exists $R \in \mathbb{R}\backslash\{0\}$ with $2R \in \operatorname{int} D$ such that all derivatives of the mapping $x \mapsto f(e^x)e^{-Rx}$ are integrable, where

$$D := \{y \in \mathbb{C} : \mathrm{E}(\exp(\mathrm{Re}\,(y) L_1)) < \infty\}.$$

Depending on the Lévy process L, less regularity of f is needed for the proofs to work. However, for ease and clarity of exposition, we do not consider the most general statements here.

4.2.3. Hedges and hedging errors

To reduce the risk arising from selling the option with payoff $f(S_T)$, we assume that the seller trades dynamically in the stock using a self-financing strategy.

Definition 4.2.3. A pair (c, ϑ) with $c \in \mathbb{R}$ and a predictable S-integrable process ϑ is called *hedge*. We refer to c as the *initial capital* and to ϑ as the *trading strategy* of the hedge.

The discounted wealth process of a hedge (c, ϑ) is $\left(c + \int_0^t \vartheta_s \, \mathrm{d}S_s\right)_{t\in[0,T]}$. We measure the performance of a hedge by its *mean squared hedging error.*

Definition 4.2.4. The *mean squared hedging error* of a hedge (c, ϑ) relative to price process S is defined by

$$\varepsilon^2(c, \vartheta, S) := \mathrm{E}\left(\left(f(S_T) - c - \int_0^T \vartheta_t \, \mathrm{d}S_t\right)^2\right).$$

4.2.3.1. Variance-optimal hedge

In incomplete market models such as the one in Section 4.2.1, there is in general no perfect hedge that leads to a vanishing mean squared hedging error. In this situation, it is natural to look for the hedge $(v, \varphi) \in \mathbb{R} \times \Theta$ with minimal mean squared hedging error, where Θ is an appropriate set of *admissible trading strategies*. This approach is called *variance-optimal hedging* and was studied intensely in the literature, cf., e.g., [Pha00, Sch01, ČernýK07] and the references therein. In the present context of geometric Lévy models, the set of admissible strategies is given by

$$\Theta = \left\{\vartheta \text{ predictable process} : \mathrm{E}\left(\int_0^T \vartheta_t^2 S_{t-}^2 \, \mathrm{d}t\right) < \infty\right\},$$

c.f. [Sch94, HKK06]. The variance-optimal hedge (v,φ) is determined in [HKK06]. The variance-optimal trading strategy satisfies the feedback equation

$$\varphi_t = \xi(t,S_{t-}) + \frac{\Lambda}{S_{t-}}\left(H(t,S_{t-}) - v - \int_0^{t-} \varphi_s \,\mathrm{d}S_s\right), \quad t\in[0,T], \tag{4.2}$$

with deterministic functions $\xi, H : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$ and a constant $\Lambda > 0$, which are to be found in Theorem 4.5.4 below. The variance-optimal initial capital v is given by $v = H(0,S_0)$. Function H is sometimes referred to as *mean value function*. By (4.2), we can express the value of the variance-optimal trading strategy at time $t\in[0,T]$ as a function of the state variables $t, S_{t-}, \int_0^{t-}\varphi_s\,\mathrm{d}S_s$, i.e.,

$$\varphi_t = \varphi\left(t, S_{t-}, \int_0^{t-}\varphi_s\,\mathrm{d}S_s\right), \quad t\in[0,T],$$

where, by slight abuse of notation, the letter φ is used to denote also the function defined as

$$\varphi(t,s,g) := \xi(t,s) + \frac{\Lambda}{s}\left(H(t,s) - v - g\right), \quad t\in[0,T],\, s\in\mathbb{R}_+,\, g\in\mathbb{R}.$$

The third state variable $\int_0^{t-}\varphi_s\,\mathrm{d}S_s$ represents the past financial gains of the investor from strategy φ. For fixed $t\in[0,T]$, $s\in\mathbb{R}_+$, and $g\in\mathbb{R}$, we refer to $\varphi(t,s,g)$ as the *variance-optimal hedge ratio in* (t,s,g).

4.2.3.2. Pure hedge

For reasonable model parameters, Λ is small, and hence the contribution of the feedback term is typically modest, and it vanishes completely if S is a martingale. Therefore, it makes sense to consider also the simpler *pure hedge* (v,ξ) defined as

$$\xi_t = \xi(t,S_{t-}), \quad t\in[0,T],$$

involving the variance-optimal initial capital v and the function ξ from (4.2). For fixed $t\in[0,T]$ and $s\in\mathbb{R}_+$, we call $\xi(t,s)$ the *pure hedge ratio in* (t,s).

In the present Lévy setup, the trading strategy ξ coincides with the so-called *locally risk-minimizing hedge* in the sense of [Sch91].

4.2.3.3. Black-Scholes hedge

Due to its relevance in practice, we also consider the Black-Scholes hedge applied to the Lévy model of Section 4.2.1. To this end, consider a standard Brownian motion $\overline{W}$ on a filtered probability space $(\overline{\Omega}, \overline{\mathscr{F}}, (\overline{\mathscr{F}}_t)_{t\in\mathbb{R}_+}, \overline{P})$, where the filtration shall be generated by $\overline{W}$. Furthermore, consider the discounted stock price process $\overline{S}$ given by

$$\overline{S}_t = S_0 e^{\mu t + \sigma \overline{W}_t}, \quad t\in\mathbb{R}_+, \tag{4.3}$$

whose parameters

$$\mu := \mathrm{E}\left(\log \frac{S_1}{S_0}\right) \quad \text{and} \quad \sigma := \sqrt{\mathrm{Var}\left(\log \frac{S_1}{S_0}\right)} \tag{4.4}$$

are chosen such that the first two moments of $\overline{L}_t := \log\left(\frac{\overline{S}_t}{S_0}\right)$ coincide with those of the logarithmic return process L from (4.1). At time $t \in [0,T]$, the unique arbitrage-free discounted price of the contingent claim with maturity T and discounted payoff $f(\overline{S}_T)$ in this model is given by $C(t,\overline{S}_t)$, where the function $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$C(t,s) = E_{\overline{Q}}\left(f(\overline{S}_T)\middle|\overline{S}_t = s\right), \quad t \in [0,T], s \in \mathbb{R}_+. \tag{4.5}$$

Here, $\overline{Q}$ denotes the unique probability measure $\overline{Q} \sim \overline{P}$ such that $\overline{S}$ is a $\overline{Q}$-martingale. Moreover, the function C is continuously differentiable with respect to the second variable s, and

$$C(0,S_0) + \int_0^T \frac{\partial}{\partial s} C(u,\overline{S}_u)\,\mathrm{d}\overline{S}_u = f(\overline{S}_T).$$

Hence, $(C(0,S_0), \frac{\partial}{\partial s}C(I,\overline{S}))$ is a perfect hedge for $f(\overline{S}_T)$ in the Black-Scholes model with discounted underlying process $\overline{S}$. In the context of our Lévy model of Section 4.2.1, we use the initial capital $C(0,S_0)$ and the function $\frac{\partial}{\partial s}C$ to define a hedge (c,ψ) given by

$$\begin{aligned} c &= C(0,S_0), \\ \psi_t &= \psi(t,S_{t-}) \end{aligned} \tag{4.6}$$

with $\psi(t,s) := \frac{\partial}{\partial s}C(t,s)$. This hedge could, e.g., be used by an investor who wrongly believes to trade in a Black-Scholes environment (4.3). We refer to it as the *Black-Scholes hedge applied to S* and to $\psi(t,s)$ as the *Black-Scholes hedge ratio in* $(t,s) \in [0,T] \times \mathbb{R}_+$. The numerical illustration of [DGMK+13] indicates that (c,ψ) is a reasonable proxy to the variance-optimal hedge for $f(S_T)$ and geometric Lévy process S.

Remark 4.2.5. If the Lévy process under consideration is a Brownian motion with drift, then variance-optimal, pure, and Black-Scholes hedge coincide, i.e.,

$$(v,\varphi) = (v,\xi) = (c,\psi).$$

Moreover, the mean squared hedging error of all three hedges vanishes. Finally, the mean value function coincides with the Black-Scholes pricing function in this case, i.e., $H(t,s) = C(t,s)$.

4.2.4. Lévy model as perturbed Black-Scholes model

We employ our perturbation approach from Chapter 2 in order to compute approximations in the sense of Principle 2.2.1 to several quantities. This requires a curve connecting the stock price process (4.1) under consideration with geometric Brownian motion. We specify such a curve, lying in the space of geometric Lévy processes, in this section.

We define processes L^λ via

$$L_t^\lambda := \left(1 - \frac{1}{\lambda}\right)\mu t + \lambda L_{\frac{t}{\lambda^2}} \quad \text{for } \lambda \in (0,1] \text{ and } t \in \mathbb{R}_+ \tag{4.7}$$

with L from Section 4.2.1 and μ, σ as in (4.4). Observe that L^λ is again a Lévy process for all $\lambda \in (0,1]$, which satisfies

$$\mathrm{E}\left(L_t^\lambda\right) = t\mu \quad \text{and} \quad \mathrm{Var}\left(L_t^\lambda\right) = t\sigma^2 \quad \text{for all } \lambda \in (0,1] \text{ and all } t \in \mathbb{R}_+. \tag{4.8}$$

Equation (4.7) does not make sense for $\lambda = 0$, but we obtain Brownian motion in the limit:

Lemma 4.2.6. *For $\lambda \to 0$, the family of Lévy processes $(L^\lambda)_{\lambda\in(0,1]}$ converges in law with respect to the Skorokhod topology (cf. [JS03, Section VI.1] for more details) to a Brownian motion with drift μ and volatility σ, i.e.,*

$$L^\lambda \xrightarrow{\mathscr{D}} \mu I + \sigma W \quad \textit{as} \quad \lambda \to 0,$$

where I denotes the identity process and W is a standard Brownian motion.

PROOF. Cf. Section 4.5.5. □

We denote the limiting process by L^0, i.e.,

$$L_t^0 := \mu t + \sigma W_t, \quad t \in \mathbb{R}_+. \tag{4.9}$$

The family of Lévy processes L^λ, $\lambda \in [0,1]$, gives rise to a family of discounted stock price processes S^λ, $\lambda \in [0,1]$, namely

$$S_t^\lambda := S_0 e^{L_t^\lambda} \quad \text{for } \lambda \in [0,1] \text{ and } t \in \mathbb{R}_+,$$

where $S_0 > 0$ denotes the initial stock price from (4.1). Note that the process S^0 coincides in law with the Black-Scholes stock price $\overline{S}$ introduced in Section 4.2.3.3.

4.2.5. Quantities to approximate

Our goal is to provide approximations to

1. the mean value function $H(t,s)$, and in particular
2. the initial capital $v = H(0,S_0)$ of the variance-optimal hedge from Section 4.2.3.1,
3. the pure hedge ratio $\xi(t,s)$ from Section 4.2.3.2,
4. the variance-optimal hedge ratio $\varphi(t,s,g)$ from Section 4.2.3.1,
5. the mean squared hedging error $\varepsilon^2(v,\xi,S)$ of the pure hedge,
6. the mean squared hedging error $\varepsilon^2(v,\varphi,S)$ of the variance-optimal hedge,

7. the mean squared hedging error $\varepsilon^2(c, \psi, S)$ of the Black-Scholes hedge from Section 4.2.3.3.

In order to employ the approach outlined in Chapter 2, we have to make sure that all the above quantities are well defined relative to S^λ, $\lambda \in [0,1]$.

Lemma 4.2.7. *The quantities listed above are well defined relative to S^λ for all $\lambda \in [0,1]$. Put differently, Assumptions 4.2.1 and 4.2.2 continue to hold and the objects from Section 4.2.3 are well defined if we replace S by S^λ.*

PROOF. Cf. Section 4.5.4. □

Let us specify in our context the notion of *second-order approximation* put forward in Principle 2.2.1.

Definition 4.2.8. Let Q denote one of the quantities listed above in the Lévy model (4.1) of interest. Moreover, let $q(\lambda)$, $\lambda \in [0,1]$, denote the corresponding quantity with respect to S^λ, and assume that $\lambda \mapsto q(\lambda)$ is twice continuously differentiable on $[0,1]$. We call

$$\mathfrak{A}(Q) := \mathfrak{A}_0(Q) + \mathfrak{A}_1(Q) + \frac{1}{2}\mathfrak{A}_2(Q)$$

with

$$\mathfrak{A}_0(Q) := q(0), \quad \mathfrak{A}_1(Q) := q'(0), \quad \mathfrak{A}_2(Q) := q''(0)$$

second-order approximation to Q (relative to the curve S^λ, $\lambda \in [0,1]$).

4.3. Approximations to hedges and hedging errors

In this section, we provide the approximations in the sense of Definition 4.2.8 to the quantities listed in Section 4.2.5. They involve two main ingredients: moments of the logarithmic return process $L = \log\left(\frac{S}{S_0}\right)$ and option price sensitivities in the limiting Black-Scholes model S^0.

4.3.1. Components of the approximations

4.3.1.1. Moments of the Lévy process

For the first four moments of the logarithmic return process L in (4.1), we obtain

$$\mathrm{E}(L_t) = \mu t, \qquad \mathrm{Var}(L_t) = \sigma^2 t,$$

$$\mathrm{Skew}(L_t) = \mathrm{Skew}(L_1)\frac{1}{\sqrt{t}}, \qquad \mathrm{ExKurt}(L_t) = \mathrm{ExKurt}(L_1)\frac{1}{t}.$$

Here, $\mathrm{Skew}(Y)$ and $\mathrm{ExKurt}(Y)$ denote skewness and excess kurtosis of a random variable Y, i.e.,

$$\mathrm{Skew}(Y) := \frac{\mathrm{E}\left((Y-\mathrm{E}(Y))^3\right)}{\sqrt{\mathrm{Var}(Y)}^3} \quad \text{and} \quad \mathrm{ExKurt}(Y) := \frac{\mathrm{E}\left((Y-\mathrm{E}(Y))^4\right)}{\sqrt{\mathrm{Var}(Y)}^4} - 3$$

if $\mathrm{E}\left(Y^4\right) < \infty$. Due to the scaling property in time, we refer to μ, σ, $\mathrm{Skew}(L_1)$, and $\mathrm{ExKurt}(L_1)$ as *drift*, *volatility*, *skewness rate*, and *excess kurtosis rate* of the logarithmic return process L.

4.3.1.2. Cash greeks in the Black-Scholes model

Applying the reasoning of Section 4.2.3.3 to the discounted stock price process S^0, we see that the unique arbitrage-free discounted price at time $t \in [0,T]$ of the option with discounted payoff $f(S^0_T)$ in the model S^0 is given by $C(t, S^0_t)$ with function $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ from (4.5).

Lemma 4.3.1. *The function $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, $(t,s) \mapsto C(t,s)$, from* (4.5) *is infinitely differentiable with respect to the second variable s for $t \in [0,T)$.*

PROOF. Cf. Lemma 3.4.1. □

For $n \in \mathbb{N}$, the quantity $\frac{\partial^n}{\partial s^n} C(t, S^0_t)$ represents the n-th order sensitivity of the option price with respect to changes in the stock price at time $t \in [0,T)$. Such sensitivities are often referred to as *greeks*. Here, we consider so-called *cash greeks*, where the sensitivity is multiplied by the corresponding power of the stock price (cf. also Section 3.4).

Definition 4.3.2. For $n \in \mathbb{N}$, we set

$$D_n(t,s) := s^n \frac{\partial^n}{\partial s^n} C(t,s), \quad t \in [0,T), s \in \mathbb{R}_+,$$

with function $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, $(t,s) \mapsto C(t,s)$, from (4.5).

4.3.2. Approximations to hedges

We begin with the approximation to the variance-optimal initial capital.

Theorem 4.3.3 (Initial capital)**.** *1. The second-order approximation in the sense of Definition 4.2.8 to the mean-value function appearing in* (4.2) *is*

$$\mathfrak{A}\left(H(t,s)\right) = \mathfrak{A}_0\left(H(t,s)\right) + \mathfrak{A}_1\left(H(t,s)\right) + \frac{1}{2}\mathfrak{A}_2\left(H(t,s)\right)$$

with

$$\begin{aligned}
\mathfrak{A}_0\left(H(t,s)\right) &= C(t,s), \\
\mathfrak{A}_1\left(H(t,s)\right) &= \mathrm{Skew}\left(L_1\right) \sigma^3 (T-t) \sum_{k=2}^{3} a_k D_k(t,s), \\
\mathfrak{A}_2\left(H(t,s)\right) &= \mathrm{Skew}\left(L_1\right)^2 \sigma^4 (T-t) \left(b_2 D_2(t,s) + \sigma^2 (T-t) \sum_{k=2}^{6} c_k D_k(t,s) \right) \\
&\quad + \mathrm{ExKurt}\left(L_1\right) \sigma^4 (T-t) \sum_{k=2}^{4} d_k D_k(t,s),
\end{aligned}$$

where $C(t,s)$ denotes the Black-Scholes pricing function from (4.5)*, $D_k(t,s)$ are cash greeks as in Definition 4.3.2, and*

$$
\begin{aligned}
a_2 &= \frac{1}{2} - \frac{1}{2}m, & a_3 &= \frac{1}{6}, & b_2 &= m - \frac{1}{6},\\
c_2 &= \frac{1}{2} - \frac{1}{3}m + \frac{1}{2}m^2, & c_3 &= \frac{13}{6} - 3m + m^2, & c_4 &= \frac{7}{4} - \frac{3}{2}m + \frac{1}{4}m^2,\\
c_5 &= \frac{5}{12} - \frac{1}{6}m, & c_6 &= \frac{1}{36}, & d_2 &= \frac{7}{12} - \frac{3}{2}m,\\
d_3 &= \frac{1}{2} - \frac{1}{3}m, & d_4 &= \frac{1}{12}, & m &= \frac{\mu + \frac{1}{2}\sigma^2}{\sigma^2}.
\end{aligned}
$$

2. *The second-order approximation to the initial capital v of both variance-optimal and pure hedge is given by*

$$\mathfrak{A}(v) = \mathfrak{A}(H(0,S_0)).$$

PROOF. Cf. Section 4.5.6. □

We proceed with the approximation to the pure hedge ratio.

Theorem 4.3.4 (Pure hedge)**.** *For the second-order approximation*

$$\mathfrak{A}(\xi(t,s)) = \mathfrak{A}_0(\xi(t,s)) + \mathfrak{A}_1(\xi(t,s)) + \frac{1}{2}\mathfrak{A}_2(\xi(t,s))$$

in the sense of Definition 4.2.8 to the pure hedge ratio for the option with payoff $f(S_T)$, we have

$$
\begin{aligned}
\mathfrak{A}_0(\xi(t,s)) &= \psi(t,s),\\
\mathfrak{A}_1(\xi(t,s)) &= \operatorname{Skew}(L_1)\,\sigma\frac{1}{s}\left(a_2 D_2(t,s) + \sigma^2(T-t)\sum_{k=2}^{4} b_k D_k(t,s)\right),\\
\mathfrak{A}_2(\xi(t,s)) &= \operatorname{Skew}(L_1)^2\,\sigma^2\frac{1}{s}\left(c_2 D_2(t,s) + \sigma^2(T-t)\sum_{k=2}^{5} d_k D_k(t,s)\right)\\
&\quad + \operatorname{Skew}(L_1)^2\,\sigma^6(T-t)^2\frac{1}{s}\sum_{k=2}^{7} e_k D_k(t,s)\\
&\quad + \operatorname{ExKurt}(L_1)\,\sigma^2\frac{1}{s}\left(\sum_{k=2}^{3} f_k D_k(t,s) + \sigma^2(T-t)\sum_{k=2}^{5} g_k D_k(t,s)\right),
\end{aligned}
$$

where $\psi(t,s)$ denotes the Black-Scholes hedge ratio from (4.6)*, $D_k(t,s)$ are the cash greeks from*

Definition 4.3.2, and

$$
\begin{aligned}
a_2 &= \frac{1}{2}, & b_2 &= 1-m, & b_3 &= 1-\frac{1}{2}m,\\
b_4 &= \frac{1}{6}, & c_2 &= -1, & d_2 &= \frac{2}{3}+m,\\
d_3 &= \frac{17}{6}-m, & d_4 &= \frac{3}{2}-\frac{1}{2}m, & d_5 &= \frac{1}{6},\\
e_2 &= 1-2m+m^2, & e_3 &= 7-10m+\frac{7}{2}m^2, & e_4 &= \frac{55}{6}-9m+2m^2,\\
e_5 &= \frac{23}{6}-\frac{7}{3}m+\frac{1}{4}m^2, & e_6 &= \frac{7}{12}-\frac{1}{6}m, & e_7 &= \frac{1}{36},\\
f_2 &= \frac{3}{2}, & f_3 &= \frac{1}{3}, & g_2 &= \frac{7}{6}-3m,\\
g_3 &= \frac{25}{12}-\frac{5}{2}m, & g_4 &= \frac{5}{6}-\frac{1}{3}m, & g_5 &= \frac{1}{12},\\
m &= \frac{\mu+\frac{1}{2}\sigma^2}{\sigma^2}.
\end{aligned}
$$

PROOF. Cf. Section 4.5.6. □

To formulate the approximation to the variance-optimal hedge ratio, we need an auxiliary result on the approximation to the mean-variance ratio Λ from Section 4.2.3.1.

Lemma 4.3.5 (Mean-variance ratio)**.** *The second-order approximation in the sense of Definition 4.2.8 to the quantity Λ from Section 4.2.3.1 is given by*

$$\mathfrak{A}(\Lambda) = \mathfrak{A}_0(\Lambda) + \mathfrak{A}_1(\Lambda) + \frac{1}{2}\mathfrak{A}_2(\Lambda),$$

where

$$
\begin{aligned}
\mathfrak{A}_0(\Lambda) &= \frac{\mu+\frac{1}{2}\sigma^2}{\sigma^2},\\
\mathfrak{A}_1(\Lambda) &= \operatorname{Skew}(L_1)\,\sigma\left(\frac{1}{6}-\mathfrak{A}_0(\Lambda)\right),\\
\mathfrak{A}_2(\Lambda) &= \operatorname{Skew}(L_1)^2\sigma^2\left(2\mathfrak{A}_0(\Lambda)-\frac{1}{3}\right)+\frac{1}{6}\operatorname{ExKurt}(L_1)\,\sigma^2\left(\frac{1}{2}-7\mathfrak{A}_0(\Lambda)\right).
\end{aligned}
$$

PROOF. Cf. Section 4.5.6. □

The approximation to the variance-optimal hedge ratio is a combination of the previously obtained approximations. To this end, we write

$$\varphi(t,s,g) = \xi(t,s) + \chi(t,s,g)$$

with

$$\chi(t,s,g) := \frac{\Lambda}{s}\left(H(t,s)-v-g\right), \quad t\in[0,T],\, s\in\mathbb{R}_+,\, g\in\mathbb{R},$$

where v denotes the variance-optimal initial capital, cf. Section 4.2.3.1.

Theorem 4.3.6 (Variance-optimal hedge)**.** *The second-order approximation in the sense of Definition 4.2.8 to the variance-optimal hedge ratio $\varphi(t,s,g)$ is of the form*

$$\mathfrak{A}\left(\varphi(t,s,g)\right) = \mathfrak{A}\left(\xi(t,s)\right) + \mathfrak{A}\left(\chi(t,s,g)\right),$$

where

$$\mathfrak{A}\left(\chi(t,s,g)\right) = \mathfrak{A}_0\left(\chi(t,s,g)\right) + \mathfrak{A}_1\left(\chi(t,s,g)\right) + \frac{1}{2}\mathfrak{A}_2\left(\chi(t,s,g)\right)$$

with

$$\begin{aligned}
\mathfrak{A}_0\left(\chi(t,s,g)\right) &= \frac{\mathfrak{A}_0\left(\Lambda\right)}{s}\left(C(t,s) - C(0,S_0) - g\right),\\
\mathfrak{A}_1\left(\chi(t,s,g)\right) &= \frac{\mathfrak{A}_1\left(\Lambda\right)}{s}\left(C(t,s) - C(0,S_0) - g\right) + \frac{\mathfrak{A}_0\left(\Lambda\right)}{s}\left(\mathfrak{A}_1\left(H(t,s)\right) - \mathfrak{A}_1\left(v\right)\right),\\
\mathfrak{A}_2\left(\chi(t,s,g)\right) &= \frac{\mathfrak{A}_2\left(\Lambda\right)}{s}\left(C(t,s) - C(0,S_0) - g\right) + 2\frac{\mathfrak{A}_1\left(\Lambda\right)}{s}\left(\mathfrak{A}_1\left(H(t,s)\right) - \mathfrak{A}_1\left(v\right)\right)\\
&\quad + \frac{\mathfrak{A}_0\left(\Lambda\right)}{s}\left(\mathfrak{A}_2\left(H(t,s)\right) - \mathfrak{A}_2\left(v\right)\right).
\end{aligned}$$

Here, the approximations to $\xi(t,s)$, Λ, and $H(t,s)$ are to be found in Theorem 4.3.4, Lemma 4.3.5, and Theorem 4.3.3. $C(t,s)$ denotes the Black-Scholes pricing function from (4.5) *for the option under consideration.*

PROOF. Cf. Section 4.5.6. □

Observe that in the above approximations, the zero-order term is always given by the respective quantity in the limiting Black-Scholes model S^0. By Remark 4.2.5, all three hedges under consideration coincide in the Black-Scholes case. Hence, the zero-order approximations of initial capital and hedge ratio are given by the Black-Scholes price resp. the Black-Scholes hedge ratio. The second-order approximations from Theorems 4.3.3, 4.3.4, and 4.3.6 thus provide model-robust corrections of the Black-Scholes initial capital and the Black-Scholes hedging strategy. Our numerical study in Section 4.4 (cf. Tables 4.1 and 4.2) shows that these corrections are excellent for a wide range of market models and payoffs.

4.3.3. Approximations to hedging errors

Theorem 4.3.7 (Variance-optimal hedging error)**.** *The second-order approximation in the sense of Definition 4.2.8 to the mean squared hedging error of the variance-optimal hedge (v,φ) is given by*

$$\begin{aligned}
\mathfrak{A}\left(\varepsilon^2(v,\varphi,S)\right) &= \frac{1}{2}\mathfrak{A}_2\left(\varepsilon^2(v,\varphi,S)\right)\\
&= \frac{1}{4}\sigma^4\left(\mathrm{ExKurt}\left(L_1\right) - \mathrm{Skew}\left(L_1\right)^2\right)\mathrm{E}\left(\int_0^T e^{-\frac{(\mu+\frac{1}{2}\sigma^2)^2}{\sigma^2}(T-t)} D_2(t,S_t^0)^2\,\mathrm{d}t\right)
\end{aligned}$$

with the cash greek $D_2(t,s)$ as in Definition 4.3.2.

PROOF. Cf. Section 4.5.6 □

Theorem 4.3.8 (Hedging error of the pure hedge)**.** *The second-order approximation in the sense of Definition 4.2.8 to the mean squared hedging error of the pure hedge (v,ξ) is given by*

$$\begin{aligned}\mathfrak{A}\left(\varepsilon^2(v,\xi,S)\right) &= \frac{1}{2}\mathfrak{A}_2\left(\varepsilon^2(v,\xi,S)\right)\\ &= \frac{1}{4}\sigma^4\left(\operatorname{ExKurt}(L_1)-\operatorname{Skew}(L_1)^2\right)\mathrm{E}\left(\int_0^T D_2(t,S_t^0)^2\,\mathrm{d}t\right)\end{aligned}$$

with the cash greek $D_2(t,s)$ as in Definition 4.3.2.

PROOF. Cf. Section 4.5.6. □

Remark 4.3.9. Note that the second-order approximations to both hedging errors differ only by the exponential dampening factor $\exp\left(-(\mu+\frac{1}{2}\sigma^2)^2\sigma^{-2}(T-t)\right)$, which appears due to the feedback term of the variance-optimal trading strategy. If the limiting Black-Scholes stock price process S^0 is a martingale, we have $\mu+\frac{1}{2}\sigma^2=0$, which implies that both approximations coincide.

Remark 4.3.10. [BKL00] study mean squared hedging errors in complete diffusion models when the replicating trading strategy of a European option is implemented discretely at time points spaced by Δt. Applied to the Black-Scholes model S^0, their findings yield that the mean squared hedging error $\varepsilon^2(c,\psi^\Delta,S^0)$ of the Black-Scholes hedge (c,ψ^Δ) implemented discretely, i.e,

$$\psi_t^\Delta = \psi_{\lfloor \frac{t}{\Delta t}\rfloor \Delta t}, \quad t\in[0,T],$$

is given by

$$\varepsilon^2(c,\psi^\Delta,S^0) = \frac{1}{2}\sigma^4\Delta t\mathrm{E}\left(\int_0^T D_2(t,S_t^0)^2\,\mathrm{d}t\right)+o(\Delta t) \quad \text{as } \Delta t\to 0.$$

Comparison with Theorem 4.3.8 suggests that, to the leading order, the risk of the pure hedge applied continuously in the Lévy model S coincides with the risk from discrete delta hedging in the Black-Scholes model S^0 with time step

$$\Delta t = \frac{1}{2}\left(\operatorname{ExKurt}(L_1)-\operatorname{Skew}(L_1)^2\right),$$

which might therefore be called the *time step equivalent of jumps*. E.g., taking $\operatorname{Skew}(L_1)=\frac{0.1}{\sqrt{250}}$ and $\operatorname{ExKurt}(L_1)=\frac{10}{250}$ as in our the numerical examples in Section 4.4, we have

$$\frac{1}{2}\left(\operatorname{ExKurt}(L_1)-\operatorname{Skew}(L_1)^2\right)\approx\frac{5}{250}.$$

Intuitively speaking, hedging continuously in the presence of jumps of the asset price approximately amounts to the same risk as weekly rebalanced delta hedging in a Black-Scholes market.

The approximation to the hedging error of the Black-Scholes hedge applied to S, given by the next theorem, is a bit more involved.

Theorem 4.3.11 (Hedging error of the Black-Scholes hedge)**.** *The second-order approximation in the sense of Definition 4.2.8 to the mean squared hedging error of the Black-Scholes hedge* (c,ψ) *applied to S, as defined in Section 4.2.3.3, is given by*

$$\begin{aligned}\mathfrak{A}\left(\varepsilon^2(c,\psi,S)\right) &= \frac{1}{2}\mathfrak{A}_2\left(\varepsilon^2(c,\psi,S)\right)\\ &= \mathfrak{A}\left(\varepsilon^2(v,\xi,S)\right) + \frac{1}{36}\mathrm{Skew}\left(L_1\right)^2\sigma^6 A(0,S_0)^2\\ &\quad + \mathrm{Skew}\left(L_1\right)^2 \mathrm{E}\left(\int_0^T\left(\frac{1}{2}\sigma^2 D_2(t,S_t^0)+\frac{1}{6}\sigma^4 B(t,S_t^0)\right)^2\mathrm{d}t\right),\end{aligned}$$

where

$$A(t,s) := \begin{cases}(T-t)\left(D_3(t,s)+3D_2(t,s)\right) & \text{if } \mu+\frac{1}{2}\sigma^2=0,\\ \frac{\widetilde{A}(t,se^{(\mu+\frac{1}{2}\sigma^2)(T-t)})-\widetilde{A}(t,s)}{\mu+\frac{1}{2}\sigma^2} & \text{if } \mu+\frac{1}{2}\sigma^2\neq 0,\end{cases} \tag{4.10}$$

$$B(t,s) := \begin{cases}(T-t)\left(D_4(t,s)+6D_3(t,s)+6D_2(t,s)\right) & \text{if } \mu+\frac{1}{2}\sigma^2=0,\\ \frac{\widetilde{B}(t,se^{(\mu+\frac{1}{2}\sigma^2)(T-t)})-\widetilde{B}(t,s)}{\mu+\frac{1}{2}\sigma^2} & \text{if } \mu+\frac{1}{2}\sigma^2\neq 0,\end{cases} \tag{4.11}$$

for

$$\begin{aligned}\widetilde{A}(t,s) &:= D_2(t,s)+D_1(t,s)-D_0(t,s),\\ \widetilde{B}(t,s) &:= D_3(t,s)+3D_2(t,s).\end{aligned}$$

Here, $D_k(t,s)$ *is as in Definition 4.3.2, and the approximation* $\mathfrak{A}\left(\varepsilon^2(v,\xi,S)\right)$ *to the mean squared hedging error of the pure hedge is provided by Theorem 4.3.8.*

PROOF. Cf. Section 4.5.6. □

Remark 4.3.12. 1. Theorems 4.3.7, 4.3.8, and 4.3.11 show that in general

$$\mathfrak{A}\left(\varepsilon^2(v,\varphi,S)\right) \le \mathfrak{A}\left(\varepsilon^2(v,\xi,S)\right) \le \mathfrak{A}\left(\varepsilon^2(c,\psi,S)\right),$$

i.e., to the leading order, the error of the pure hedge is bigger than that of the variance-optimal hedge but smaller than that of the Black-Scholes hedge. However, if the law of L_1 is not skewed, then the leading order errors of pure and Black-Scholes hedge coincide. As our numerical study in Section 4.4 shows, the difference between these two approximations is also in the case of non-zero skewness typically negligible since $\mathrm{Skew}\left(L_1\right)^2$ is comparably small. If, in addition to vanishing skewness, the limiting discounted Black-Scholes stock price process S^0 is a martingale, the approximations to the hedging errors of variance-optimal, pure, and Black-Scholes hedge coincide, cf. Remark 4.3.9.

2. The functions $A(t,s)$ and $B(t,s)$ in (4.10) and (4.11) are pointwise continuous in $\mu+\frac{1}{2}\sigma^2$ at $\mu+\frac{1}{2}\sigma^2=0$.

Remark 4.3.13. Lemma 4.5.2 below provides an integral representation of cash greeks in the Black-Scholes model via the Laplace transform approach (cf. also Chapter 3). This permits efficient evaluation of the products of cash greeks in Theorems 4.3.7–4.3.11 by numerical integration.

4.4. Numerical comparison

In this section, we examine the accuracy of the approximations from Section 4.3 by numerical examples. To this end, we compare exact and approximate initial capital, initial hedge ratio, and root mean squared hedging error of the variance-optimal hedge. Moreover, we compare the exact and approximate root mean squared hedging error of the Black-Scholes hedge. We perform our study for European call options in three different parametric Lévy models.

4.4.1. Market models

As parametric market models for the discounted stock, we consider Merton's jump-diffusion (JD) model with normal jumps [Mer76], the normal inverse Gaussian (NIG) model [BN98], and the variance gamma (VG) model [MS90] for various parameter choices.

As initial stock price, we always set $S_0 = 100$. Moreover, we fix the parameters of all models such that

$$\begin{aligned}
\mu &= \mathrm{E}(L_1) = -0.08, \\
\sigma^2 &= \mathrm{Var}(L_1) = 0.4^2, \\
\mathrm{Skew}(L_1) &= \frac{0.1}{\sqrt{250}}.
\end{aligned}$$

The excess kurtosis rate $\mathrm{ExKurt}(L_1)$ is chosen as $2/250$, $5/250$, $10/250$, respectively. All these choices are well within the range of empirically plausible values, cf., e.g., [CGMY02, Table 4]. Note that skewness rate and excess kurtosis rate are reported such that one directly recovers the values on a daily basis, assuming 250 trading days per year. Moreover, our choice is such that $\mu + \frac{1}{2}\sigma^2 = 0$, i.e., the stock has the risk free rate of return. Hence, the mean squared hedging errors of variance-optimal and pure hedge coincide in this situation, cf. Remark 4.3.9.

NIG and VG are models with four parameters, and so the specification of the first four moments of logarithmic returns leaves no degree of freedom. The JD model, however, has five parameters. In order to eliminate the additional degree of freedom, the parameters are chosen such that the volatility arising from the jump component explains 70% of the overall volatility of logarithmic returns.

In order to calculate the exact values of the quantities of interest, we use the formulas from Section 4.5.4 and perform standard numerical quadrature.

4.4.2. Option payoff function

We consider European calls with strike $K = 95$, 100, or 105, respectively, and maturity $T = 1/12$, $1/4$, or $1/2$, measured in years. The corresponding payoff function $f(s) = (s-K)^+$ allows for an integral representation as in (4.12), given by

$$f(s) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} \mathrm{d}z, \quad s \in \mathbb{R}_+,$$

for arbitrary $R > 1$, cf. Example 3.2.3. Strictly speaking, the kinked payoff function of the European call does not meet the smoothness requirement of Assumption 4.2.2. Nevertheless, the approximate formulas from Section 4.3 are well defined in this situation, as one easily shows by use of Lemma 4.5.7. Hence, we can and will use them in our numerical comparison.

4.4.3. Hedges and hedging errors

In any of the above cases, we compute the initial capital v, the initial hedge ratio $\varphi(0, S_0, v)$, and the square root $\sqrt{\varepsilon^2(v, \varphi, S)}$ of the mean squared hedging error of the variance-optimal hedge. These are compared to the respective approximations from Theorems 4.3.3, 4.3.6, and 4.3.7. Moreover, we report the corresponding Black-Scholes price $c = C(0, S_0)$ and the initial Black-Scholes hedge ratio $\psi(0, S_0)$. Finally, we compute the square root $\sqrt{\varepsilon(c, \psi, S)}$ of the exact mean squared hedging error of the Black-Scholes hedge and compare it to the approximation from Theorem 4.3.11.

4.4.4. Discussion of the numerical results

Table 4.1 shows the exact and approximate variance-optimal initial capital as well as the Black-Scholes price for different models and payoffs. Table 4.2 reports the exact and approximate variance-optimal hedge ratio for $t = 0$ as well as the initial Black-Scholes hedge ratio. For both quantities, the exact values mostly coincide across the jump models, and the approximation is precise up to the last digit. For high excess kurtosis and short maturity, the performance of the approximations is slightly worse, but also the improvement compared to the mere Black-Scholes value becomes more pronounced.

Table 4.3 shows exact and approximate values for the square root of the mean squared hedging error of the variance-optimal hedge. In brackets, we report the exact resp. approximate square root of the mean squared hedging error of the Black-Scholes hedge. We observe that the difference between the approximations to both strategies seems negligible for practical purposes. Moreover, the approximations tend to slightly overestimate the exact values. The performance becomes worse for shorter time to maturity and higher excess kurtosis rate. In the case of the variance-optimal hedge, e.g., for $K = 100$ and $\mathrm{ExKurt}(\mathrm{L}_1) = 2/250$, the relative deviation of the approximate value from the average exact value over all jump models amounts to 6.7% for $T = 1/12$ and to 2.4% for $T = 1/2$. For $K = 100$ and $\mathrm{ExKurt}(\mathrm{L}_1) = 10/250$, the relative deviation accounts for 18% in the case $T = 1/12$ and for 6.0% in the case $T = 1/2$. As already pointed out in [DGMK$^+$13], we see from the respective hedging errors that the mere Black-Scholes hedge is a satisfying proxy to the variance-optimal hedge.

As mentioned above, the approximations to the error of variance-optimal and pure hedge from Theorems 4.3.7 and 4.3.8 coincide in our study since we choose $\mu + \frac{1}{2}\sigma^2 = 0$. However, numerical experiments that are not shown here indicate that for typical parameter choices, the difference between these two approximations is negligible (in the magnitude of less than 1%) also if $\mu + \frac{1}{2}\sigma^2 \neq 0$. Hence, for practical purposes, the simplest of our formulas – the one from Theorem 4.3.8 – should be used to approximately quantify the error of either pure, variance-optimal, or Black-Scholes hedge.

ExKurt(L$_1$)	K	$T=\frac{1}{12}$					$T=\frac{1}{4}$					$T=\frac{1}{2}$				
		JD	NIG	VG	BS	Approx	JD	NIG	VG	BS	Approx	JD	NIG	VG	BS	Approx
$\frac{2}{250}$	95	7.406	7.406	7.406	7.424	7.406	10.511	10.511	10.511	10.520	10.511	13.641	13.641	13.641	13.644	13.641
	100	4.589	4.589	4.589	4.604	4.589	7.961	7.961	7.961	7.966	7.961	11.247	11.247	11.247	11.246	11.247
	105	2.631	2.631	2.631	2.634	2.631	5.907	5.907	5.907	5.906	5.907	9.202	9.202	9.202	9.197	9.202
$\frac{5}{250}$	95	7.385	7.386	7.385	7.424	7.384	10.496	10.496	10.496	10.520	10.496	13.631	13.631	13.631	13.644	13.631
	100	4.562	4.564	4.562	4.604	4.562	7.946	7.946	7.946	7.966	7.946	11.237	11.237	11.237	11.246	11.237
	105	2.614	2.614	2.614	2.634	2.613	5.894	5.895	5.894	5.906	5.894	9.194	9.194	9.194	9.197	9.194
$\frac{10}{250}$	95	7.351	7.355	7.351	7.424	7.348	10.472	10.473	10.472	10.520	10.472	13.614	13.615	13.614	13.644	13.614
	100	4.520	4.524	4.518	4.604	4.517	7.922	7.923	7.922	7.966	7.921	11.221	11.222	11.221	11.246	11.221
	105	2.586	2.588	2.586	2.634	2.583	5.874	5.875	5.874	5.906	5.874	9.180	9.180	9.180	9.197	9.180

Table 4.1.: Exact and approximate variance-optimal initial capital and Black-Scholes price for $\mu=-0.08$, $\sigma=0.4$, $\mathrm{Skew}(\mathrm{L}_1)=\frac{0.1}{\sqrt{250}}$ and varying excess kurtosis rate $\mathrm{Exkurt}(\mathrm{L}_1)$, strike K, and maturity T

ExKurt(L$_1$)	K	$T=\frac{1}{12}$					$T=\frac{1}{4}$					$T=\frac{1}{2}$				
		JD	NIG	VG	BS	Approx	JD	NIG	VG	BS	Approx	JD	NIG	VG	BS	Approx
$\frac{2}{250}$	95	0.696	0.696	0.696	0.692	0.696	0.642	0.642	0.642	0.639	0.642	0.628	0.628	0.628	0.627	0.628
	100	0.528	0.528	0.528	0.523	0.528	0.543	0.543	0.543	0.540	0.543	0.558	0.558	0.558	0.556	0.558
	105	0.363	0.363	0.363	0.358	0.363	0.446	0.446	0.446	0.443	0.446	0.490	0.490	0.490	0.488	0.490
$\frac{5}{250}$	95	0.697	0.697	0.697	0.692	0.697	0.643	0.643	0.643	0.639	0.643	0.629	0.629	0.629	0.627	0.629
	100	0.530	0.530	0.530	0.523	0.530	0.544	0.544	0.544	0.540	0.544	0.559	0.559	0.559	0.556	0.559
	105	0.367	0.367	0.367	0.358	0.367	0.447	0.447	0.447	0.443	0.448	0.491	0.491	0.491	0.488	0.491
$\frac{10}{250}$	95	0.699	0.699	0.699	0.692	0.699	0.645	0.645	0.645	0.639	0.645	0.631	0.631	0.631	0.627	0.631
	100	0.534	0.534	0.534	0.523	0.535	0.546	0.546	0.546	0.540	0.547	0.561	0.561	0.561	0.556	0.561
	105	0.373	0.372	0.373	0.358	0.373	0.450	0.450	0.450	0.443	0.450	0.493	0.493	0.493	0.488	0.493

Table 4.2.: Exact and approximate initial variance-optimal hedge ratio $\varphi(0,S_0,v)$ as well as initial Black-Scholes hedge ratio $\psi(0,S_0)$ for $\mu=-0.08$, $\sigma=0.4$, $\mathrm{Skew}(\mathrm{L}_1)=\frac{0.1}{\sqrt{250}}$ and varying excess kurtosis rate $\mathrm{Exkurt}(\mathrm{L}_1)$, strike K, and maturity T

ExKurt(L_1)	K	$T=\frac{1}{12}$ JD	NIG	VG	Approx	$T=\frac{1}{4}$ JD	NIG	VG	Approx	$T=\frac{1}{2}$ JD	NIG	VG	Approx
	95	0.756 (0.764)	0.746 (0.753)	0.760 (0.768)	0.808 (0.810)	0.812 (0.817)	0.807 (0.811)	0.814 (0.819)	0.841 (0.843)	0.827 (0.830)	0.824 (0.827)	0.829 (0.832)	0.847 (0.849)
$\frac{2}{250}$	100	0.837 (0.846)	0.828 (0.836)	0.840 (0.851)	0.891 (0.893)	0.859 (0.865)	0.855 (0.860)	0.861 (0.867)	0.889 (0.892)	0.865 (0.869)	0.864 (0.867)	0.868 (0.870)	0.887 (0.889)
	105	0.818 (0.829)	0.813 (0.823)	0.821 (0.833)	0.871 (0.873)	0.871 (0.878)	0.869 (0.874)	0.874 (0.880)	0.902 (0.905)	0.886 (0.890)	0.885 (0.889)	0.889 (0.892)	0.908 (0.910)
	95	1.145 (1.162)	1.121 (1.138)	1.153 (1.173)	1.280 (1.281)	1.256 (1.269)	1.243 (1.255)	1.260 (1.274)	1.332 (1.333)	1.289 (1.299)	1.282 (1.291)	1.292 (1.301)	1.341 (1.343)
$\frac{5}{250}$	100	1.270 (1.290)	1.246 (1.266)	1.279 (1.303)	1.411 (1.413)	1.330 (1.345)	1.319 (1.332)	1.334 (1.350)	1.408 (1.410)	1.350 (1.361)	1.344 (1.353)	1.352 (1.363)	1.404 (1.405)
	105	1.243 (1.267)	1.227 (1.250)	1.249 (1.277)	1.379 (1.380)	1.350 (1.366)	1.341 (1.356)	1.354 (1.371)	1.429 (1.430)	1.383 (1.395)	1.378 (1.388)	1.385 (1.397)	1.438 (1.439)
	95	1.530 (1.560)	1.492 (1.522)	1.553 (1.591)	1.811 (1.812)	1.730 (1.754)	1.703 (1.726)	1.738 (1.767)	1.884 (1.885)	1.792 (1.812)	1.776 (1.794)	1.797 (1.819)	1.898 (1.899)
$\frac{10}{250}$	100	1.701 (1.738)	1.660 (1.698)	1.728 (1.775)	1.997 (1.998)	1.834 (1.862)	1.808 (1.835)	1.842 (1.874)	1.993 (1.994)	1.878 (1.900)	1.863 (1.883)	1.883 (1.906)	1.986 (1.987)
	105	1.675 (1.720)	1.644 (1.688)	1.689 (1.743)	1.951 (1.952)	1.863 (1.894)	1.842 (1.872)	1.870 (1.905)	2.022 (2.023)	1.924 (1.948)	1.912 (1.934)	1.929 (1.955)	2.034 (2.035)

Table 4.3.: Exact and approximate square root of mean squared hedging error of the variance-optimal hedge for $\mu=-0.08$, $\sigma=0.4$, $\mathrm{Skew}(L_1)=\frac{0.1}{\sqrt{250}}$ and varying excess kurtosis rate $\mathrm{Exkurt}(L_1)$, strike K, and maturity T; the values in brackets denote the exact resp. approximate mean squared hedging error of the Black-Scholes hedge with volatility parameter $\sigma=0.4$

4.5. Proofs

In this section, we present the proofs of the assertions from Sections 4.2 and 4.3.

4.5.1. Outline

The derivations of the approximations to initial capitals, hedge ratios, and hedging errors all follow the same pattern: for the respective object, we dispose of a deterministic representation in terms of an integral representation of the payoff function f (stated in Section 4.5.2). Formally, the quantity of interest $q(\lambda)$ relative to S^λ, $\lambda \in [0,1]$, can be written as

$$q(\lambda) = \int h(\lambda, z)\,\mathrm{d}z.$$

To obtain the second-order approximation to $q(1)$ in the sense of Definition 4.2.8, we will perform three steps:

1. Ensure that $h(\lambda, z)$ is twice partially differentiable with respect to λ and that integration with respect to z and differentiation with respect to λ can be interchanged.

2. Compute $h(0,z)$, $\frac{\partial}{\partial\lambda}h(0,z)$, and $\frac{\partial^2}{\partial\lambda^2}h(0,z)$.

3. Express the integrals over the derivatives from Step 2 in terms of moments of L_1 and cash greeks of the limiting Black-Scholes model S^0.

4.5.2. Integral representation of the payoff function

Lemma 4.5.1. *There exist $R \in \mathbb{R}\backslash\{0\}$ with $2R \in \operatorname{int} D$ and a function $p : (R + i\mathbb{R}) \to \mathbb{C}$ such that the payoff function f from Section 4.2.2 admits the representation*

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^z\, p(z)\,\mathrm{d}z, \quad s \in \mathbb{R}_+. \tag{4.12}$$

Moreover, $x \mapsto |R+ix|^n\,|p(R+ix)|$ is integrable for all $n \in \mathbb{N}$.

PROOF. By Assumption 4.2.2, there exists $R \in \mathbb{R}\backslash\{0\}$ with $2R \in \operatorname{int} D$ such that all derivatives of $x \mapsto f(e^x)e^{-Rx}$ are integrable on $\mathbb{R}$. Then Lemma 3.3.1 directly yields the assertion. □

From now on, we fix R as in Lemma 4.5.1.

4.5.3. Integral representation of cash greeks in the Black-Scholes model

From the integral representation (4.12) of the payoff function, we readily obtain an integral representation for cash greeks in the Black-Scholes model S^0.

Lemma 4.5.2. *$D_n(t,s)$ in Definition 4.3.2 can be written as*

$$D_n(t,s) = \int_{R-i\infty}^{R+i\infty} \left(\prod_{i=0}^{n-1} (z-i) \right) s^z e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)} p(z)\,\mathrm{d}z$$

for any $n \in \mathbb{N}$, $t \in [0,T)$, and $s \in \mathbb{R}_+$.

PROOF. Cf. Lemma 3.4.3. □

4.5.4. Exact representations of hedges and hedging errors

By making use of Representation (4.12), [HKK06] derive representations of variance-optimal and pure hedge and of the associated mean squared hedging errors. Their formulas are expressed in terms of the integral representation of the payoff function and the *cumulant generating function* of the driving Lévy process.

Definition 4.5.3. For $\lambda \in [0,1]$, the *cumulant generating function* of L^λ is the unique continuous function $\kappa^\lambda : D^\lambda \to \mathbb{C}$ such that

$$\mathrm{E}\left(e^{zL_t^\lambda}\right) = e^{t\kappa^\lambda(z)}$$

for all $t \in \mathbb{R}_+$ and all $z \in D^\lambda := \left\{ y \in \mathbb{C} : \mathrm{E}\left(e^{\mathrm{Re}(y)L_1^\lambda}\right) < \infty \right\}$.

For existence and uniqueness of the cumulant generating function, cf. [Sat99, Lemma 7.6].

Theorem 4.5.4 ([HKK06])**.** *1. Let $\lambda \in [0,1]$. For the stock price process S^λ and the contingent claim with payoff $f(S_T^\lambda)$, the variance-optimal initial capital v^λ and the variance-optimal trading strategy φ^λ are given by*

$$v^\lambda = H^\lambda(0,S_0) \tag{4.13}$$

and

$$\varphi_t^\lambda = \varphi^\lambda(t, S_{t-}^\lambda, G_{t-}^\lambda), \quad t \in [0,T],$$

for the function $\varphi^\lambda : [0,T] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ given by

$$\varphi^\lambda(t,s,g) := \xi^\lambda(t,s) + \frac{\Lambda^\lambda}{s}(H^\lambda(t,s) - v^\lambda - g). \tag{4.14}$$

Here, the functions $H^\lambda, \xi^\lambda : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, the process G^λ, and the constant Λ^λ are defined by

$$\begin{aligned}
\bar{\kappa}^\lambda(y,z) &:= \kappa^\lambda(y+z) - \kappa^\lambda(y) - \kappa^\lambda(z), \\
\gamma^\lambda(z) &:= \frac{\bar{\kappa}^\lambda(z,1)}{\bar{\kappa}^\lambda(1,1)}, \\
\eta^\lambda(z) &:= \kappa^\lambda(z) - \kappa^\lambda(1)\gamma^\lambda(z), \qquad (4.15)\\
\Lambda^\lambda &:= \frac{\kappa^\lambda(1)}{\bar{\kappa}^\lambda(1,1)}, \\
H^\lambda(t,s) &:= \int_{R-i\infty}^{R+i\infty} s^z e^{\eta^\lambda(z)(T-t)}\, p(z)\,\mathrm{d}z, \qquad (4.16)\\
\xi^\lambda(t,s) &:= \int_{R-i\infty}^{R+i\infty} s^{z-1}\gamma^\lambda(z) e^{\eta^\lambda(z)(T-t)}\, p(z)\,\mathrm{d}z, \qquad (4.17)\\
G_t^\lambda &:= \int_0^t \varphi_s^\lambda\,\mathrm{d}S_s^\lambda.
\end{aligned}$$

2. *The corresponding mean squared hedging error of the variance-optimal hedge*

$$\varepsilon^2(v^\lambda, \varphi^\lambda, S^\lambda) = \mathrm{E}\left(\left(f(S_T^\lambda) - v^\lambda - \int_0^T \varphi_t^\lambda\,\mathrm{d}S_t^\lambda\right)^2\right)$$

is given by

$$\varepsilon^2(v^\lambda, \varphi^\lambda, S^\lambda) = \int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty}\int_0^T J_1^\lambda(t,y,z)\,p(y)p(z)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z,$$

where

$$\begin{aligned}
\rho_j^\lambda(y,z) &:= \eta^\lambda(y) + \eta^\lambda(z) - j\frac{\kappa^\lambda(1)^2}{\bar{\kappa}^\lambda(1,1)}, \quad j \in \{0,1\}, \qquad (4.18)\\
\beta^\lambda(y,z) &:= \bar{\kappa}^\lambda(y,z) - \frac{\bar{\kappa}^\lambda(y,1)\bar{\kappa}^\lambda(z,1)}{\bar{\kappa}^\lambda(1,1)}, \\
J_j^\lambda(t,y,z) &:= S_0^{y+z}\beta^\lambda(y,z)e^{\kappa^\lambda(y+z)t+\rho_j^\lambda(y,z)(T-t)}, \quad j \in \{0,1\}. \qquad (4.19)
\end{aligned}$$

3. *The mean squared hedging error*

$$\varepsilon^2(v^\lambda, \xi^\lambda, S^\lambda) = \mathrm{E}\left(\left(f(S_T^\lambda) - v^\lambda - \int_0^T \xi_t^\lambda\,\mathrm{d}S_t^\lambda\right)^2\right)$$

of the pure hedge (i.e., the hedge using the initial capital v^λ from (4.13) *and the trading strategy $\xi_t^\lambda = \xi^\lambda(t, S_{t-}^\lambda)$ from* (4.17)*) is given by*

$$\varepsilon^2(v^\lambda, \xi^\lambda, S^\lambda) = \int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty}\int_0^T J_0^\lambda(t,y,z)\,p(y)p(z)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z$$

with the function J_0^λ from (4.19).

PROOF. By construction, it is obvious that for all $\lambda \in [0,1]$, Assumptions 4.2.1 and 4.2.2 hold as well for L^λ. With the integral representation of f by Lemma 4.5.1 at hand, we can apply [HKK06, Theorems 3.1 and 3.2], which shows Assertions 1 and 2. Assertion 3 follows from the proof of [HKK06, Theorem 3.2]. Note that we can choose the same parameter R for all $\lambda \in [0,1]$ because $D = D^1 \subseteq D^\lambda$. □

Remark 4.5.5. Theorem 4.5.4 holds under the much milder assumption that $x \mapsto |p(R+ix)|$ is integrable on $\mathbb{R}$ (cf. [HKK06, Section 2]).

Following the approach of [HKK06], [DGMK+13] study the error of suboptimal strategies in geometric Lévy models. The next theorem restates their main result in a special case, which is sufficient for our purposes.

Theorem 4.5.6 ([DGMK+13])**.** *Let $\lambda \in [0,1]$. For the stock price process S^λ, consider the initial capital $d^\lambda \in \mathbb{R}$ and the trading strategy*

$$\vartheta_t^\lambda = \vartheta^\lambda(t, S_{t-}^\lambda), \quad t \in [0,T],$$

with the function $\vartheta^\lambda : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$\vartheta^\lambda(t,s) := \int_{R-i\infty}^{R+i\infty} s^{z-1} z e^{\frac{1}{2}\nu^2 z(z-1)(T-t)} p(z)\,\mathrm{d}z \tag{4.20}$$

for $\nu > 0$. The resulting mean squared hedging error

$$\varepsilon^2(d^\lambda, \vartheta^\lambda, S^\lambda) = \mathrm{E}\left(\left(f(S_T^\lambda) - d^\lambda - \int_0^T \vartheta_t^\lambda\,\mathrm{d}S_t^\lambda\right)^2\right)$$

is given by

$$\varepsilon^2(d^\lambda, \vartheta^\lambda, S^\lambda) = (w^\lambda - d^\lambda)^2 + \int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty}\int_0^T J_2^\lambda(t,y,z)\,p(y)p(z)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z, \tag{4.21}$$

where

$$\begin{aligned}
\alpha^\lambda(z,t) &:= \left(1 - \kappa^\lambda(1)\int_t^T z e^{\kappa^\lambda(z)(s-T)} e^{\frac{1}{2}\nu^2 z(z-1)(T-s)}\,\mathrm{d}s\right) e^{\kappa^\lambda(z)(T-t)}, && (4.22)\\
w^\lambda &:= \int_{R-i\infty}^{R+i\infty} S_0^z \alpha^\lambda(z,0)\,p(z)\,\mathrm{d}z, && (4.23)\\
h^\lambda(t,y,z) &:= \bar{\kappa}^\lambda(y,z)\alpha^\lambda(y,t)\alpha^\lambda(z,t) - \bar{\kappa}^\lambda(y,1)\alpha^\lambda(y,t) z e^{\frac{1}{2}\nu^2 z(z-1)(T-t)}\\
&\quad - \bar{\kappa}^\lambda(z,1)\alpha^\lambda(z,t) y e^{\frac{1}{2}\nu^2 y(y-1)(T-t)}\\
&\quad + \bar{\kappa}^\lambda(1,1) yz e^{(\frac{1}{2}\nu^2(z(z-1)+y(y-1))(T-t)}, && (4.24)\\
J_2^\lambda(t,y,z) &:= S_0^{y+z} e^{\kappa^\lambda(y+z)t} h^\lambda(t,y,z). && (4.25)
\end{aligned}$$

PROOF. As noted above, for all $\lambda \in [0,1]$, Assumptions 4.2.1 and 4.2.2 hold as well for L^λ by construction. With the integral representation of f by Lemma 4.5.1 at hand, we obtain that ϑ^λ is a Δ-strategy in the sense of [DGMK+13, Definition 3.1]. The assertion then follows from Theorem 4.2 of the same paper. As noted in the proof of Theorem 4.5.4, we can choose the same parameter R for all $\lambda \in [0,1]$. □

The above representations for the quantities of interest enable us to give the

PROOF OF LEMMA 4.2.7. By Theorem 4.5.4, the Quantities 1–6 from Section 4.2.5 are obviously well defined relative to S^λ, $\lambda \in [0,1]$, since Assumptions 4.2.1 and 4.2.2 hold for all L^λ, $\lambda \in [0,1]$, by construction. From its definition and Lemma 4.5.2, we obtain that the Black-Scholes trading strategy ψ^λ applied to S^λ admits the representation

$$\psi_t^\lambda = \int_{R-i\infty}^{R+i\infty} \left(S_{t-}^\lambda\right)^{z-1} z e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)} p(z)\,\mathrm{d}z, \quad t \in [0,T].$$

The resulting mean squared hedging error is given by Theorem 4.5.6. □

4.5.5. Technicalities

Lemma 4.5.7. *For $n \in \mathbb{N}$, $m \in \{0,1,2\}$, and $t \in [0,T)$, the mappings*

$$z \mapsto \left| z^n e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)} \right| \quad \textit{and} \quad z \mapsto \int_0^T \left| z^m e^{\frac{1}{2}\sigma^2 z(z-1)(T-s)} \right| \mathrm{d}s$$

are bounded on $R + i\mathbb{R}$.

PROOF. Observe that

$$\mathrm{Re}\left(\frac{1}{2}\sigma^2 z(z-1)\right) = \frac{1}{2}\sigma^2 \left(R^2 - R - \mathrm{Im}(z)^2\right) = \frac{1}{2}\sigma^2(2R^2 - R) - \frac{1}{2}\sigma^2 |z|^2$$

for $z \in R + i\mathbb{R}$. The first assertion then follows from the fact that the mapping $x \mapsto x^n e^{-ax^2}$, $a > 0$, is bounded on $\mathbb{R}_+$ for all $n \in \mathbb{N}$. The second assertion follows by simple integration. □

Proposition 4.5.9 below will be the building block for the forthcoming computations; it states the derivatives of $\kappa^\lambda(z)$ with respect to λ. To obtain these, we work with $\kappa^\lambda(z)$ in terms of its Lévy-Khintchine triplet, cf. [Sat99, Theorem 8.1] for more details.

Lemma 4.5.8. *Let (b,c,F) denote the Lévy-Khintchine triplet of the Lévy process L with respect to the truncation function $x \mapsto x 1_{[-1,1]}(x)$. For all $n \in \{2,\dots 5\}$, there is measurable $g_n : \mathbb{R} \to \mathbb{R}_+$ such that*

$$\left| x^n e^{\xi z x} \right| \le g_n(x) \quad \textit{for all } x \in \mathbb{R}, \xi \in [0,1], \textit{ and } z \in \{0, R, 2R\} + i\mathbb{R}$$

and such that

$$\int g_n(x) F(\mathrm{d}x) < \infty.$$

PROOF. Let $\xi \in [0,1]$ and $z \in \{0,R,2R\} + i\mathbb{R}$. In the case $R \geq 0$, we have $\xi \mathrm{Re}(z) \geq 0$, and hence

$$\left|x^n e^{\xi z x}\right| \leq |x^n| \, 1_{(-\infty,0)}(x) + x^n e^{2Rx} 1_{[0,\infty)}(x) =: g_n(x).$$

Then,

$$\int g_n(x) F(dx) \leq \int_{\{x<0\}} |x|^n F(dx) + \int_{\{x\geq 0\}} x^n e^{2Rx} F(dx). \tag{4.26}$$

The first integral on the right-hand side is finite by Assumption 4.2.1, [Sat99, Theorem 25.3], and because F is a Lévy measure, which integrates $x \mapsto x^2$ in a neighborhood of 0. To handle the second integral, choose $\varepsilon > 0$ such that $2R+\varepsilon \in D$, which is possible because $2R \in \mathrm{int} D$ by Assumption 4.2.2. Since the exponential function grows faster than any polynomial, there exists $A_\varepsilon > 0$ such that $x^n e^{2Rx} \leq e^{(2R+\varepsilon)x}$ for all $x \geq A_\varepsilon$. Hence, we have

$$\int_{\{x\geq 0\}} x^n e^{2Rx} F(dx) \leq \int_{\{0\leq x< A_\varepsilon \vee 1\}} x^n e^{2Rx} F(dx) + \int_{\{x\geq A_\varepsilon \vee 1\}} e^{(2R+\varepsilon)x} F(dx).$$

The first integral on the right-hand side is finite since F is a Lévy measure, and the second one is finite by [Sat99, Theorem 25.3] since $2R+\varepsilon \in D$. Altogether, we have shown that both integrals in (4.26) are finite, which proves the assertion for $R \geq 0$. The case $R < 0$ is treated along the same lines. □

Proposition 4.5.9. *For the family of cumulant generating functions $\kappa^\lambda(z)$ of L^λ, $\lambda \in [0,1]$, understood as a mapping $\kappa : [0,1] \times (\{0,R,2R\} + i\mathbb{R}) \to \mathbb{C}$, $(\lambda, z) \mapsto \kappa^\lambda(z)$, we have the following: κ is twice partially differentiable with respect to λ, and κ, $\frac{\partial}{\partial\lambda}\kappa$, $\frac{\partial^2}{\partial\lambda^2}\kappa$ are continuous. In particular,*

$$\begin{aligned}
\kappa^0(z) &= \mu z + \frac{1}{2}\sigma^2 z^2, \\
\frac{\partial}{\partial\lambda}\kappa^0(z) &= \frac{1}{6}\mathrm{E}\left((L_1-\mu)^3\right) z^3, \\
\frac{\partial^2}{\partial\lambda^2}\kappa^0(z) &= \frac{1}{12}\left(\mathrm{E}\left((L_1-\mu)^4\right) - 3\sigma^4\right) z^4.
\end{aligned}$$

We have the estimates

$$\left|\frac{\partial^n}{\partial\lambda^n}\kappa^\lambda(z)\right| \leq c_1(1+|z|^{3+n}) \quad \textit{for all } (\lambda,z) \in [0,1] \times (\{0,R,2R\}+i\mathbb{R}),\ n \in \{0,1,2\},$$

where $c_1 > 0$ is a constant that does not depend on λ, z, n.

PROOF. Form the definition of L^λ in (4.7), it follows directly that its cumulant generating function κ^λ is given in terms of κ^1 by

$$\kappa^\lambda(z) = \left(1 - \frac{1}{\lambda}\right)\mu z + \frac{1}{\lambda^2}\kappa^1(\lambda z), \quad \lambda \in (0,1],\ z \in \{0,R,2R\} + i\mathbb{R} \subset D.$$

For L^0 as defined in (4.9), it is immediate that $\kappa^0(z) = \mu z + \frac{1}{2}\sigma^2 z^2$ for $z \in \{R, 2R\} + i\mathbb{R}$. Denote by (b, c, F) the Lévy-Khintchine triplet of $L = L^1$ with respect to the truncation function $x \mapsto x1_{[-1,1]}(x)$. By [Sat99, Theorem 25.17], we have that

$$\kappa^1(z) = bz + \frac{1}{2}cz^2 + \int e^{zx} - 1 - zx1_{[-1,1]}(x)\, F(dx), \quad z \in \{0, R, 2R\} + i\mathbb{R}. \tag{4.27}$$

Moreover,

$$\mathrm{E}\,(L_1) = \mu = b + \int x1_{[-1,1]^C}(x)\, F(dx) \tag{4.28}$$

by [Sat99, Example 25.12]. Combining these two representations, we obtain that

$$\kappa^\lambda(z) = \mu z + \frac{1}{2}cz^2 + \int \frac{1}{\lambda^2}\left(e^{\lambda zx} - 1 - \lambda zx\right) F(dx), \quad \lambda \in (0,1], z \in \{0, R, 2R\} + i\mathbb{R}.$$

Making use of the Taylor expansion with integral remainder term

$$e^{\lambda zx} = 1 + \lambda zx + \frac{1}{2}(\lambda zx)^2 + \frac{1}{2}(\lambda zx)^3 \int_0^1 e^{s\lambda zx}(1-s)^2\,\mathrm{d}s,$$

we deduce that

$$\kappa^\lambda(z) = \mu z + \frac{1}{2}z^2\left(c + \int x^2\, F(dx)\right) + \frac{1}{2}z^3 \int\int_0^1 \lambda x^3 e^{s\lambda zx}(1-s)^2\,\mathrm{d}s\, F(dx) \tag{4.29}$$

for $\lambda \in [0,1]$, $z \in \{0, R, 2R\} + i\mathbb{R}$. Observe that this representation holds also for $\lambda = 0$ since by [Sat99, Example 25.12],

$$\mathrm{Var}\,(L_1) = \sigma^2 = c + \int x^2\, F(dx). \tag{4.30}$$

The integrand in (4.29) is obviously twice partially differentiable with respect to λ. Lemma 4.5.8 and Corollary C.0.3 yield that integration and differentiation can be interchanged. The straightforward calculations yield

$$\frac{\partial}{\partial\lambda}\kappa^\lambda(z) = \frac{1}{2}z^3 \int\int_0^1 x^3 e^{s\lambda zx}(1-s)^2(1 + \lambda zxs)\,\mathrm{d}s\, F(dx),$$
$$\frac{\partial^2}{\partial\lambda^2}\kappa^\lambda(z) = \frac{1}{2}z^4 \int\int_0^1 x^4 e^{s\lambda zx} s(1-s)^2(2 + \lambda zxs)\,\mathrm{d}s\, F(dx)$$

for $\lambda \in [0,1], z \in \{0, R, 2R\} + i\mathbb{R}$. The continuity of κ, $\frac{\partial}{\partial\lambda}\kappa$, and $\frac{\partial^2}{\partial\lambda^2}\kappa$ as well as their polynomial growth in z follow now from the above representations, Lemma 4.5.8, and dominated convergence. Evaluating $\frac{\partial}{\partial\lambda}\kappa^\lambda$ and $\frac{\partial^2}{\partial\lambda^2}\kappa^\lambda$ in $\lambda = 0$ yields

$$\frac{\partial}{\partial\lambda}\kappa^0(z) = \frac{1}{6}z^3 \int x^3\, F(dx) \quad \text{and} \quad \frac{\partial^2}{\partial\lambda^2}\kappa^0(z) = \frac{1}{12}z^4 \int x^4\, F(dx)$$

for $z \in \{0, R, 2R\} + i\mathbb{R}$. [Sat99, Example 25.12] derives (4.28) and (4.30) based on the relation between moments of a random variable and derivatives of its characteristic function, cf., e.g., [Sat99, Proposition 2.5(ix)]. Applying the same reasoning to the higher moments of L_1 (which has the characteristic function $u \mapsto e^{\kappa^1(iu)}$) yields after straightforward calculations using (4.27)

$$\int x^3\, F(dx) = \mathrm{E}\left((L_1 - \mu)^3\right) \quad \text{and} \quad \int x^4\, F(dx) = \mathrm{E}\left((L_1 - \mu)^4\right) - 3\sigma^4.$$

The existence of the moments is given by Assumption 4.2.1, which completes the proof. □

The previous result allows us to give the proof of Lemma 4.2.6 on the convergence of L^λ to Brownian motion as $\lambda \to 0$.

PROOF OF LEMMA 4.2.6. Proposition 4.5.9 yields directly that

$$\lim_{\lambda\to 0} e^{\kappa^\lambda(iu)} = e^{\mu iu-\frac{1}{2}\sigma^2u^2} \quad \text{for all } u \in \mathbb{R}.$$

By Lévy's continuity theorem (cf., e.g., [Sat99, Proposition 2.5(vii)]), the univariate marginals of L^λ converge to the univariate marginals of $\mu I + \sigma B$ as $\lambda \to 0$, where B denotes standard Brownian motion. By [JS03, Corollary VII.3.6], this implies convergence of the whole process, which completes the proof. □

Lemma 4.5.10. *1. The mapping*

$$(\lambda, z) \mapsto \left|e^{\kappa^\lambda(z)}\right| = e^{\mathrm{Re}\left(\kappa^\lambda(z)\right)}$$

is bounded on $[0,1]\times(\{R,2R\}+i\mathbb{R})$.

2. The mapping

$$(\lambda, z) \mapsto \left|e^{\eta^\lambda(z)}\right| = e^{\mathrm{Re}\left(\eta^\lambda(z)\right)}$$

is bounded on $[0,1]\times(R+i\mathbb{R})$.

PROOF. 1. For all $\lambda \in [0,1]$ and all $z \in \{R,2R\}+i\mathbb{R}$, we have

$$e^{\mathrm{Re}\left(\kappa^\lambda(z)\right)} = \left|e^{\kappa^\lambda(z)}\right| = \left|\mathrm{E}\left(e^{zL_1^\lambda}\right)\right| \le \mathrm{E}\left(\left|e^{zL_1^\lambda}\right|\right) = \mathrm{E}\left(e^{\mathrm{Re}(z)L_1^\lambda}\right) = e^{\kappa^\lambda(\mathrm{Re}(z))} \tag{4.31}$$

by Jensen's inequality. By Proposition 4.5.9, $(\lambda, r) \mapsto \kappa^\lambda(r)$ is bounded as continuous mapping on the compact set $[0,1]\times\{R,2R\}$. Since $\mathrm{Re}(z) \in \{R,2R\}$, the first assertion follows.

2. By [HKK06, Lemma 3.4], we have the inequality

$$\left|\gamma^\lambda(z)\right|^2 = \frac{\left|\kappa^\lambda(z+1)-\kappa^\lambda(z)-\kappa^\lambda(1)\right|^2}{\bar{\kappa}^\lambda(1,1)^2} \le \bar{\kappa}^\lambda(1,1)\left(\kappa^\lambda(2\mathrm{Re}(z)) - 2\mathrm{Re}\left(\kappa^\lambda(z)\right)\right)$$

for all $\lambda \in [0,1]$ and all $z \in R+i\mathbb{R}$. Hence,

$$\begin{aligned}\left|\kappa^\lambda(1)\gamma^\lambda(z)\right|^2 &\le \kappa^\lambda(1)^2\bar{\kappa}^\lambda(1,1)\kappa^\lambda(2R) - 2\kappa^\lambda(1)^2\bar{\kappa}^\lambda(1,1)\mathrm{Re}\left(\kappa^\lambda(z)\right)\\ &\le (c^\lambda)^2 + \frac{1}{4}\mathrm{Re}\left(\kappa^\lambda(z)\right)^2\\ &\le \left(c^\lambda + \frac{1}{2}\left|\mathrm{Re}\left(\kappa^\lambda(z)\right)\right|\right)^2,\end{aligned}$$

where

$$c^\lambda := \sqrt{\left|\kappa^\lambda(1)^2\bar{\kappa}^\lambda(1,1)\kappa^\lambda(2R)\right| + 4\left(2\kappa^\lambda(1)^2\bar{\kappa}^\lambda(1,1)\right)^2}, \quad \lambda \in [0,1].$$

This yields

$$\begin{aligned}\operatorname{Re}\left(\eta^{\lambda}(z)\right) &= \operatorname{Re}\left(\kappa^{\lambda}(z)\right) - \operatorname{Re}\left(\kappa^{\lambda}(1)\gamma^{\lambda}(z)\right) \\ &\leq \operatorname{Re}\left(\kappa^{\lambda}(z)\right) + \left|\kappa^{\lambda}(1)\gamma^{\lambda}(z)\right| \\ &\leq \operatorname{Re}\left(\kappa^{\lambda}(z)\right) + \frac{1}{2}\left|\operatorname{Re}\left(\kappa^{\lambda}(z)\right)\right| + c^{\lambda} \\ &\leq \frac{3}{2}\left|\kappa^{\lambda}(R)\right| + c^{\lambda}\end{aligned}$$

because $\operatorname{Re}\left(\kappa^{\lambda}(z)\right) \leq \kappa^{\lambda}(R)$ by (4.31). Proposition 4.5.9 yields that $\lambda \mapsto c^{\lambda}$ and $\lambda \mapsto \kappa^{\lambda}(R)$ are bounded as continuous mappings on $[0,1]$, which completes the proof. □

Lemma 4.5.11. *There exists $c_2 > 0$ such that $\bar{\kappa}^{\lambda}(1,1) > c_2$ for all $\lambda \in [0,1]$.*

PROOF. By Assumption 4.2.1 and (4.8), $\operatorname{Var}\left(L_1^{\lambda}\right) = \operatorname{Var}(L_1) > 0$ for all $\lambda \in [0,1]$. Hence,

$$\operatorname{Var}\left(e^{L_1^{\lambda}}\right) = \operatorname{E}\left(e^{2L_1^{\lambda}}\right) - \operatorname{E}\left(e^{L_1^{\lambda}}\right)^2 = e^{\kappa^{\lambda}(2)} - e^{2\kappa^{\lambda}(1)} > 0 \quad \text{for all } \lambda \in [0,1],$$

which implies $\kappa^{\lambda}(2) - 2\kappa^{\lambda}(1) = \bar{\kappa}^{\lambda}(1,1) > 0$ for all $\lambda \in [0,1]$. Since $\lambda \mapsto \bar{\kappa}^{\lambda}(1,1)$ is continuous by Proposition 4.5.9, it attains its minimum on $[0,1]$, which shows the assertion. □

4.5.6. Proofs of the main theorems

PROOF OF THEOREM 4.3.3. By Theorem 4.5.4, we have $v^{\lambda} = H^{\lambda}(0,S_0)$, $\lambda \in [0,1]$, for the function

$$H^{\lambda}(t,s) = \int_{R-i\infty}^{R+i\infty} s^z e^{\eta^{\lambda}(z)(T-t)} p(z)\,\mathrm{d}z, \quad t \in [0,T], s \in \mathbb{R}_+,$$

from (4.16). For the remainder of the proof, we fix $t \in [0,T]$ and $s \in \mathbb{R}_+$. From (4.15) and Proposition 4.5.9, we obtain

$$\eta^0(z) = \frac{1}{2}\sigma^2 z(z-1), \quad z \in R + i\mathbb{R}. \tag{4.32}$$

Lemma 4.5.2 then yields $H^0(t,s) = C(t,s)$ with $C(t,s)$ from (4.5). To prove the assertion, we show that $\lambda \mapsto H^{\lambda}(t,s)$ is twice continuously differentiable on $[0,1]$, and we will identify the derivatives in $\lambda = 0$. For fixed $z \in R + i\mathbb{R}$, elementary calculus and Proposition 4.5.9 yield that $\lambda \mapsto e^{\eta^{\lambda}(z)(T-t)}$ is twice continuously differentiable on $[0,1]$. It follows from Lemma 4.5.10(2), Proposition 4.5.9, Lemma 4.5.11, and Lemma 4.5.1 that there exists a majorant $m : (R+i\mathbb{R}) \to \mathbb{R}_+$ such that

$$\left| s^z \frac{\partial^n}{\partial \lambda^n} e^{\eta^{\lambda}(z)(T-t)} \right| \leq s^R m(z) \quad \text{for all } \lambda \in [0,1], z \in R + i\mathbb{R}, n \in \{1,2\},$$

and

$$\int_{-\infty}^{\infty} s^R m(R+ix)\,|p(R+ix)|\,\mathrm{d}x < \infty.$$

By Corollary C.0.3 and dominated convergence, $\lambda \mapsto H^\lambda(t,s)$ is hence twice continuously differentiable on $[0,1]$, and

$$\frac{\partial^n}{\partial\lambda^n} H^\lambda(t,s) = \int_{R-i\infty}^{R+i\infty} s^z \frac{\partial^n}{\partial\lambda^n} e^{\eta^\lambda(z)(T-t)} p(z)\,\mathrm{d}z \quad \text{for all } \lambda\in[0,1], n\in\{1,2\}. \tag{4.33}$$

For shorter notation, set

$$q_n(z) \coloneqq \prod_{k=0}^{n-1}(z-k), \quad z\in\mathbb{C}, n\in\mathbb{N}. \tag{4.34}$$

Using the derivatives of $\kappa^\lambda(z)$ from Proposition 4.5.9, we obtain after lengthy but straightforward calculations that for all $z\in R+i\mathbb{R}$

$$\begin{aligned}
\frac{\partial}{\partial\lambda} e^{\eta^\lambda(z)(T-t)}\Big|_{\lambda=0} &= \operatorname{Skew}(L_1)\,\sigma^3(T-t)e^{\eta^0(z)(T-t)}\sum_{k=2}^{3} a_k q_k(z),\\
\frac{\partial^2}{\partial\lambda^2} e^{\eta^\lambda(z)(T-t)}\Big|_{\lambda=0} &= \operatorname{Skew}(L_1)^2\,\sigma^4(T-t)e^{\eta^0(z)(T-t)}\left(b_2q_2(z)+\sigma^2(T-t)\sum_{k=2}^{6} c_k q_k(z)\right)\\
&\quad + \operatorname{ExKurt}(L_1)\,\sigma^4(T-t)e^{\eta^0(z)(T-t)}\sum_{k=2}^{4} d_k q_k(z)
\end{aligned}$$

with constants $a_2,a_3,\dots,d_4$ as in Theorem 4.3.3. In view of (4.32) and (4.33), the assertion follows now from the integral representation of cash greeks given in Lemma 4.5.2. □

PROOF OF THEOREM 4.3.4. Fix $t\in[0,T]$ and $s\in\mathbb{R}_+$. By definition in (4.17), we have

$$\xi^\lambda(t,s) = \int_{R-i\infty}^{R+i\infty} s^{z-1}\gamma^\lambda(z)e^{\eta^\lambda(z)(T-t)}p(z)\,\mathrm{d}z.$$

From Proposition 4.5.9, we obtain that $\gamma^0(z)=z$, and hence $\xi^0(t,s)=\frac{1}{s}D_1(t,s)=\psi(t,s)$ by (4.32) and Lemma 4.5.2. From now on, one proceeds as in the proof of Theorem 4.3.3; differentiability of the integrand and existence of the majorant follow from the same lemmas. We restrict ourselves to giving the result of the essential calculation:

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\gamma^\lambda(z)e^{\eta^\lambda(z)(T-t)}\Big|_{\lambda=0} &= \operatorname{Skew}(L_1)\,\sigma e^{\eta^0(z)(T-t)}\left(a_2q_2(z)+\sigma^2(T-t)\sum_{k=2}^{4} b_k q_k(z)\right),\\
\frac{\partial^2}{\partial\lambda^2}\gamma^\lambda(z)e^{\eta^\lambda(z)(T-t)}\Big|_{\lambda=0} &= \operatorname{Skew}(L_1)^2\,\sigma^2 e^{\eta^0(z)(T-t)}\left(c_2q_2(z)+\sigma^2(T-t)\sum_{k=2}^{5} d_k q_k(z)\right)\\
&\quad + \operatorname{Skew}(L_1)^2\,\sigma^6(T-t)^2e^{\eta^0(z)(T-t)}\sum_{k=2}^{7} e_k q_k(z))\\
&\quad + \operatorname{ExKurt}(L_1)\,\sigma^2 e^{\eta^0(z)(T-t)}\sum_{k=2}^{3} f_k q_k(z)\\
&\quad + \operatorname{ExKurt}(L_1)\,\sigma^4(T-t)e^{\eta^0(z)(T-t)}\sum_{k=2}^{5} g_k q_k(z)
\end{aligned}$$

for constants $a_2,\dots,g_5$ as in Theorem 4.3.4. The assertion follows then from Lemma 4.5.2. □

PROOF OF LEMMA 4.3.5. The assertion follows directly by elementary calculus, using the derivatives of $\kappa^\lambda(z)$ in $\lambda = 0$ from Proposition 4.5.9. $\square$

PROOF OF THEOREM 4.3.6. For fixed $t \in [0,T]$, $s \in \mathbb{R}_+$, and $g \in \mathbb{R}_+$, the assertion follows directly from the approximations to $\xi(t,s)$, Λ, $H(t,s)$, and v given in Theorem 4.3.4, Lemma 4.3.5, and Theorem 4.3.3. $\square$

PROOF OF THEOREMS 4.3.7 AND 4.3.8. For shorter notation, let $\varepsilon_0^2(\lambda) := \varepsilon^2(v^\lambda, \xi^\lambda, S^\lambda)$ and $\varepsilon_1^2(\lambda) := \varepsilon^2(v^\lambda, \varphi^\lambda, S^\lambda)$, $\lambda \in [0,1]$, be the mean squared hedging errors of pure and variance-optimal hedge relative to S^λ. In order to prove the assertion, we will show that $\lambda \mapsto \varepsilon_j^2(\lambda)$, $j \in \{0,1\}$, is twice continuously differentiable on $[0,1]$, and we will identify the derivatives in $\lambda = 0$. To this end, we use the deterministic representation of $\varepsilon_j^2(\lambda)$ from Theorem 4.5.4. Inserting κ^0 from Proposition 4.5.9 immediately yields that $\varepsilon_0^2(0) = \varepsilon_1^2(0) = 0$. For fixed $(t,y,z) \in [0,T] \times (R+i\mathbb{R}) \times (R+i\mathbb{R})$, the mapping $\lambda \mapsto J_j^\lambda(t,y,z)$ with J_j^λ from (4.19) is twice continuously differentiable on $[0,1]$ by Proposition 4.5.9 and elementary differential calculus. Moreover, by Proposition 4.5.9, Lemmas 4.5.1, 4.5.10, and 4.5.11 $\frac{\partial}{\partial\lambda} J_j^\lambda(t,y,z)$ and $\frac{\partial^2}{\partial\lambda^2} J_j^\lambda(t,y,z)$ admit a majorant $m : [0,T] \times (R+i\mathbb{R}) \times (R+i\mathbb{R}) \to \mathbb{R}_+$, more precisely,

$$\left| \frac{\partial^n}{\partial\lambda^n} J_j^\lambda(t,y,z) \right| \leq m(t,y,z) \quad \text{for all } \lambda \in [0,1],\ t \in [0,T],\ y,z \in R+i\mathbb{R},\ n \in \{1,2\}$$

such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^T m(t, R+iu, R+iv)\, |p(R+iu)|\, |p(R+iv)|\, \mathrm{d}t\, \mathrm{d}u\, \mathrm{d}v < \infty.$$

Hence, by Corollary C.0.3 and dominated convergence, $\lambda \mapsto \varepsilon_j^2(\lambda)$ is twice continuously differentiable on $[0,1]$, and

$$\frac{\partial^n}{\partial\lambda^n} \varepsilon_j^2(\lambda) = \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \int_0^T \frac{\partial^n}{\partial\lambda^n} J_j^\lambda(t,y,z)\, p(y) p(z)\, \mathrm{d}t\, \mathrm{d}y\, \mathrm{d}z \quad \text{for all } \lambda \in [0,1],\ n \in \{1,2\}.$$

By lengthy but straightforward calculations, we obtain from Proposition 4.5.9 that we have

$$\frac{\partial}{\partial\lambda} J_j^\lambda(t,y,z) \Big|_{\lambda=0} = 0, \quad j \in \{0,1\},$$

and

$$\begin{aligned} \frac{\partial^2}{\partial\lambda^2} J_j^\lambda(t,y,z) \Big|_{\lambda=0} &= \frac{1}{2}\sigma^4 \left(\mathrm{ExKurt}(L_1) - \mathrm{Skew}(L_1)^2 \right) e^{-j \frac{\left(\mu + \frac{1}{2}\sigma^2\right)^2}{\sigma^2}(T-t)} \\ &\quad \times S_0^{y+z} e^{\kappa^0(y+z)t} e^{\eta^0(y)(T-t)} e^{\eta^0(z)(T-t)} y(y-1)z(z-1). \end{aligned}$$

In order to interpret the integral over this derivative in the desired way, we use Fubini's Theorem (whose application is justified by Lemmas 4.5.1 and 4.5.10) and Lemma 4.5.2 and obtain

$$
\begin{aligned}
&\int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty}\int_0^T S_0^{y+z} e^{\kappa^0(y+z)t} e^{(\eta^0(y)+\eta^0(z))(T-t)} y(y-1)z(z-1)\,p(y)p(z)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z\\
&=\int_0^T\int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty} \mathrm{E}\left((S_t^0)^{y+z}\right) e^{(\eta^0(y)+\eta^0(z))(T-t)} y(y-1)z(z-1)\,p(y)p(z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\\
&=\mathrm{E}\left(\int_0^T \left((S_t^0)^2\int_{R-i\infty}^{R+i\infty}(S_t^0)^{y-2}e^{\eta^0(y)(T-t)}y(y-1)\,p(y)\,\mathrm{d}y\right)^2\mathrm{d}t\right)\\
&=\mathrm{E}\left(\int_0^T D_2(t,S_t^0)^2\,\mathrm{d}t\right),
\end{aligned}
$$

which completes the proof. □

PROOF OF THEOREM 4.3.11. By definition in Section 4.2.3.3 and by Lemma 4.5.2, the Black-Scholes hedge (c^λ,ψ^λ) applied to S^λ admits the integral representation

$$
\begin{aligned}
c^\lambda &= \int_{R-i\infty}^{R+i\infty} S_0)^z e^{\frac{1}{2}\sigma^2 z(z-1)T} p(z)\,\mathrm{d}z,\\
\psi_t^\lambda &= \int_{R-i\infty}^{R+i\infty} (S_{t-}^\lambda)^{z-1} z e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)} p(z)\,\mathrm{d}z, \quad t\in[0,T].
\end{aligned}
$$

Hence, Theorem 4.5.6 can be applied with $\nu=\sigma$ and $d^\lambda=c^\lambda$. Thus, we obtain a deterministic integral representation of the hedging error $\varepsilon^2(c^\lambda,\psi^\lambda,S^\lambda)$ for $\lambda\in[0,1]$. Observe that $\varepsilon^2(c^0,\psi^0,S^0)=0$. The reasoning to show the assertion is now analogous to the proof of Theorems 4.3.7 and 4.3.8. The existence of the necessary majorants and differentiability of $\lambda\mapsto\varepsilon^2(c^\lambda,\psi^\lambda,S^\lambda)$ on $[0,1]$ follow from the same lemmas. Tedious but straightforward calculations based on Proposition 4.5.9 yield $\frac{\partial}{\partial\lambda} J_2^\lambda(t,y,z)\Big|_{\lambda=0}=0$ and

$$
\begin{aligned}
\frac{\partial^2}{\partial\lambda^2} J_2^\lambda(t,y,z)\Big|_{\lambda=0} &= S_0^{y+z} e^{\kappa^0(y+z)t} e^{(\eta^0(y)+\eta^0(z))(T-t)}\\
&\quad\times\Bigg(\frac{1}{2}\sigma^4\mathrm{ExKurt}(L_1)\,y(y-1)z(z-1)+\frac{1}{18}\sigma^8\mathrm{Skew}(L_1)^2\,b(y,t)b(z,t)\\
&\qquad+\frac{1}{6}\sigma^6\mathrm{Skew}(L_1)^2\left(y(y-1)b(z,t)+z(z-1)b(y,t)\right)\Bigg)
\end{aligned}\tag{4.35}
$$

for J_2^λ from (4.25) in the case $(d^\lambda,\vartheta^\lambda)=(c^\lambda,\psi^\lambda)$ and with

$$
c(z):=\left(\mu+\frac{1}{2}\sigma^2\right)z \quad\text{and}\quad b(z,t):=(z^4-z^2)\int_t^T e^{c(z)(s-t)}\,\mathrm{d}s,\quad z\in R+i\mathbb{R},\ t\in[0,T].
$$

In the case $\mu+\frac{1}{2}\sigma^2=0$, we have $b(z,t)=(T-t)(q_4(z)+6q_3(z)+6q_2(z))$ with $q_n(z)$ as defined in (4.34). The expected time integral in the assertion is obtained as in the proof of Theorems 4.3.7 and 4.3.8. In the case $\mu+\frac{1}{2}\sigma^2\neq 0$, we have

$$
b(z,t)=(q_3(z)+3q_2(z))\left(\exp(c(z)(T-t))-1\right)\left(\mu+\frac{1}{2}\sigma^2\right)^{-1}.
$$

To see how to handle the additional term $\exp(c(z)(T-t))$, let us exemplarily consider the relevant part of the second summand in (4.35)

$$\begin{aligned}
&\int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty}\int_0^T S_0^{y+z}e^{\kappa^0(y+z)t}e^{(\eta^0(y)+\eta^0(z))(T-t)}b(y,t)b(z,t)\,p(y)p(z)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z\\
&\qquad = \mathrm{E}\left(\int_0^T\left(\int_{R-i\infty}^{R+i\infty}\left(S_t^0\right)^z e^{\eta^0(z)(T-t)}\frac{e^{c(z)(T-t)}-1}{\mu+\frac{1}{2}\sigma^2}(q_3(z)+3q_2(z))\,p(z)\,\mathrm{d}z\right)^2\mathrm{d}t\right),
\end{aligned}$$

where we used that $S_0^{y+z}e^{\kappa^0(y+z)t}=\mathrm{E}\left((S_t^0)^{y+z}\right)$ and Fubini's Theorem, whose application is justified by Lemmas 4.5.1 and 4.5.10. By Lemma 4.5.2, we obtain that

$$\begin{aligned}
&\int_{R-i\infty}^{R+i\infty}\left(S_t^0\right)^z e^{\eta^0(z)(T-t)}\frac{e^{c(z)(T-t)}-1}{\mu+\frac{1}{2}\sigma^2}(q_3(z)+3q_2(z))\,p(z)\,\mathrm{d}z\\
&=\int_{R-i\infty}^{R+i\infty}e^{\eta^0(z)(T-t)}\frac{\left(S_t^0e^{(\mu+\frac{1}{2}\sigma^2)(T-t)}\right)^z-\left(S_t^0\right)^z}{\mu+\frac{1}{2}\sigma^2}\,(q_3(z)+3q_2(z))\,p(z)\,\mathrm{d}z\\
&=\frac{D_3(t,S_t^0e^{(\mu+\frac{1}{2}\sigma^2)(T-t)})+3D_2(t,S_t^0e^{(\mu+\frac{1}{2}\sigma^2)(T-t)})-D_3(t,S_t^0)-3D_2(t,S_t^0)}{\mu+\frac{1}{2}\sigma^2}.
\end{aligned}$$

Differentiability and the interpretation of the derivatives in $\lambda=0$ of the mapping $\lambda\mapsto\left(w^\lambda-d^\lambda\right)$ are treated completely analogously to the proof of Theorem 4.3.3. Summing up all calculations, we obtain

$$\varepsilon^2(c^\lambda,\psi^\lambda,S^\lambda)\Big|_{\lambda=0}=\frac{\partial}{\partial\lambda}\varepsilon(c^\lambda,\psi^\lambda,S^\lambda)\Big|_{\lambda=0}=0$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial\lambda^2}\varepsilon(c^\lambda,\psi^\lambda,S^\lambda)\Big|_{\lambda=0}&=\frac{1}{18}\mathrm{Skew}\,(L_1)^2\,\sigma^6A(0,S_0)^2\\
&\quad+\frac{1}{2}\mathrm{ExKurt}\,(L_1)\,\sigma^4\mathrm{E}\left(\int_0^T D_2(t,S_t^0)^2\,\mathrm{d}t\right)\\
&\quad+\frac{1}{18}\mathrm{Skew}\,(L_1)^2\,\sigma^8\mathrm{E}\left(\int_0^T B(t,S_t^0)^2\,\mathrm{d}t\right)\\
&\quad+\frac{1}{3}\mathrm{Skew}\,(L_1)^2\,\sigma^6\mathrm{E}\left(\int_0^T B(t,S_t^0)D_2(t,S_t^0)\,\mathrm{d}t\right)
\end{aligned}$$

with the mappings $A,B:[0,T]\times\mathbb{R}_+\to\mathbb{R}$ as defined in (4.10), (4.11). Reordering and comparison with Theorem 4.3.8 completes the proof. □

4.6. Conclusion

We provide second-order approximations to the variance-optimal and pure hedge as well as to the mean squared hedging errors of these two strategies and the Black-Scholes hedge when the discounted stock price follows a geometric Lévy process and the payoff function is smooth. The approximations are obtained by considering the Lévy model of interest as a perturbed Black-Scholes model (cf. also Chapter 2). More specifically, our approach relies on connecting the Lévy model under consideration with the approximating Black-Scholes model by a curve in the set of stochastic processes. Its specific choice affects both structure and concrete form of the approximations. In principle, the curve could be chosen differently as long as one ends up with computable formulas that yield reasonable results in practically relevant cases. We leave the discussion of alternative curves to future research.

Qualitatively, our results show that the deviation of hedges and hedging errors from Black-Scholes is essentially determined by the third and fourth moment of logarithmic returns in the Lévy model and by Black-Scholes sensitivities (cash greeks) of the option. The fine structure of the Lévy process is less relevant. The option contributes to the hedging error primarily through its Black-Scholes gamma.

Quantitatively, for models from the literature and reasonable parameter values, numerical tests indicate that the accuracy of our approximations is excellent for initial capital and hedge ratios and reasonable for their hedging errors. Moreover, our tests suggest that the Black-Scholes strategy is a very good proxy to the variance-optimal one, and its hedging error due to the jumps of the Lévy process is essentially determined by the excess kurtosis of logarithmic stock returns. By comparison with results on discrete-time hedging, one may say that the risk of the Black-Scholes hedge in the presence of jumps is approximately the same as if the Black-Scholes delta is implemented discretely in a Black-Scholes market at time steps

$$\Delta t = \frac{1}{2}\Big(\mathrm{ExKurt}\,(\text{log-returns}) - (\mathrm{Skew}\,(\text{log-returns}))^2\Big).$$

5. Approximate no-arbitrage option pricing in stochastic volatility models

5.1. Introduction

In the celebrated Black-Scholes model [BS73], the dynamics of the underlying relative to the unique risk-neutral measure is given by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad S_0 > 0, \tag{5.1}$$

where $S_0 > 0$ is the initial price of the underlying, $r \in \mathbb{R}$ is the riskless interest rate, $\sigma > 0$, and W is a standard Brownian motion. Note that r and S_0 are quantities observable on the market, and hence only the *volatility parameter* σ is to be determined. In the Black-Scholes model, the price of a European call option with maturity $T > 0$ and strike $K > 0$ is given by

$$c(T,K) := e^{-rT}\mathrm{E}\left((S_T - K)^+\right). \tag{5.2}$$

For fixed model parameters r and S_0, there is a one-to-one relation between the price $c(T,K)$ and the volatility parameter σ. Hence, the observation of a single call option price on the market allows to determine σ. One calls this volatility parameter fitting to the observed option price *implied volatility*. If the market is consistent with the Black-Scholes model, then all call option prices can be explained by (5.2) basing on this implied volatility. Put differently, all implied volatilities obtained from call options with different strikes and maturities coincide if the the Black-Scholes model is valid.

However, this contradicts empirical facts on real markets [DFW98, Reb99, CDF02]: for options on stocks or foreign exchange rates, the implied volatility strongly depends on the strike. This dependence is either decreasing – the so-called *skew* – or U-shaped – the so-called *smile*. Moreover, the shape of implied volatilities is also dependent on the maturity of the option: the skew and smile get less pronounced for longer maturities. This phenomenon is called the *term structure* of implied volatilities.

In order to capture these characteristics of empirical implied volatility surfaces, a plethora of generalizations or alternatives to the Black-Scholes model (5.1) has been considered in the literature. Most of these models are also motivated by the aim to reflect certain empirical facts of asset price time series in a better way than the Black-Scholes model. However, we focus on the level of risk-neutral modelling in the following.

One natural generalization of (5.1) is obtained by replacing the constant volatility parameter σ by a stochastic process $(\sigma_t)_{t\in\mathbb{R}_+}$, i.e.,

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}W_t, \quad S_0 > 0, \tag{5.3}$$

which leads to so-called *stochastic volatility models*. In bivariate diffusion models of this kind, $(\sigma_t)_{t\in\mathbb{R}_+}$ is as well driven by a standard Brownian motion that is possibly correlated with W. The first model of this manner (without correlation of Brownian motions) was considered by [HW87]. A popular choice among practitioners is the Heston model [Hes93]. Bivariate diffusion stochastic volatility models reasonably capture the shape of implied volatility surfaces for medium and longer maturities but not for short-termed options since the stochasticity of volatility needs some time to manifest itself. In the model proposed by Barndorff-Nielsen & Shephard [BNS01, BNNS02, BNS03], the squared volatility process in (5.3) is driven by a Lévy-Ornstein-Uhlenbeck process, i.e., the volatility process exhibits jumps in contrast to bivariate diffusion models.

Another canonical generalization of (5.1) is to keep with the independent and stationary increments of logarithmic returns but to waive continuity of the paths. This leads to *geometric Lévy* models, where the underlying price process is given by

$$S_t = S_0 e^{rt+L_t}, \quad t \in \mathbb{R}_+,$$

for the initial underlying price $S_0 > 0$ and a Lévy process L. This class of models has been studied intensely in the literature, c.f., e.g., [EK95, Ryd97, BN95, MS90, MCC98, Rai00, CGMY02] and the monographs [Sch03, CT03]. For fixed and in particular short-termed maturities, geometric Lévy models reasonable fit the implied volatility smile or skew, while the term structure is not captured adequately due to the homogeneity of the increments of L.

Time-changed Lévy models, introduced by [CGMY03], try to overcome this drawback by modelling the stock price process as

$$S_t = S_0 e^{rt+L_{\int_0^t y_s\,\mathrm{d}s}}, \quad t \in \mathbb{R}_+,$$

where $S_0 > 0$ is the initial underlying price, L is a Lévy process, and y is a suitable non-negative process independent of L. A relation between the movements of y and the logarithmic underlying price $\log(S)$ corresponding to correlation in bivariate diffusion models can be incorporated by extending the model to

$$S_t = S_0 e^{rt+L_{\int_0^t y_s\,\mathrm{d}s}+\rho(y_t-y_0)}, \quad t \in \mathbb{R}_+,$$

for $\rho \in \mathbb{R}_+$. Time-changed Lévy models allow for a reasonable fit to implied volatility surfaces of short-termed and long-termed options.

For a more detailed discussion of stochastic volatility models, comparison, and parametric examples, we refer the reader to [CT03, Chapter 15].

For many parametric specifications of the mentioned model classes, the characteristic function of the logarithmic underlying price $\log(S_T)$ is available in closed form, which opens the door to the Laplace transform approach to price European options in a numerically efficient way, cf. Chapter 3. However, the related integral representation of the option price provides little insight into the determining factors of the model on option prices, e.g., which features of a certain model class determine the smile or skew. Moreover, the numerical quadrature required in the Laplace transform approach is sometimes too slow when it comes to calibration or risk-management problems. For these reasons, there are multiple contributions in the literature deriving approximate representations of option prices, aiming at more insight into the structure of prices or faster numerical computation. Most of the articles focus on specific model classes or parametric models. We review some contributions in detail in Section 5.6.

We contribute to this stream of the literature by deriving an approximate formula for the price of a European option in a very general framework, encompassing bivariate diffusion models with correlation, geometric Lévy, and time-changed Lévy models. To this end, we employ our general perturbation approach from Chapter 2, i.e., we interpret a complex stock price model as a perturbation of a simple Black-Scholes model and compute corresponding second-order corrections for the option price. As in the case of quadratic hedging from Chapter 4, our formula is given in terms of moments of components of the stock price process and sensitivities of the option price in the Black-Scholes model.

This chapter is organized as follows. In Section 5.2, we present our mathematical setup. In particular, we propose a curve connecting the stock price model of interest with geometric Brownian motion. We demonstrate in Section 5.3 that our framework encompasses several important stock price models from the literature. In Section 5.4, we present and comment our approximate option pricing formula, and we derive a related approximation to implied volatility. Detailed numerical tests of our approximation in four parametric models are performed in Section 5.5. In particular, we compare the performance of our formula with approximations from the literature. We discuss these contributions in Section 5.6. All lengthy or technical proofs are delegated to Section 5.7. We conclude in Section 5.8.

5.2. Mathematical setup

5.2.1. Market model and option

We consider a market consisting of two traded assets, a bond and a non-dividend paying stock. The price process B of the bond is given by

$$B_t = e^{rt}, \quad t \in \mathbb{R}_+,$$

for a deterministic interest rate $r \geq 0$. In what follows, we will always work with discounted quantities relative to the numéraire B, which will therefore play no role in the remainder.

Always operating on a probability space $(\Omega, \mathscr{F}, P)$, we assume the discounted price process of the stock S to be given by a *stochastic volatility model* in the sense of the following

Definition 5.2.1. A tuple (S_0, L, V, U) is called *stochastic volatility model* if the following holds.

(i) $S_0 \in \mathbb{R}_+$.

(ii) L is an intrinsic Lévy process with càdlàg paths such that $\mathrm{E}\left(e^{L_1}\right) = 1$ and $\mathrm{Var}\left(L_1\right) > 0$.

(iii) V is a strictly increasing, continuous stochastic process with $V_0 = 0$.

(iv) U is a special semimartingale with respect to the filtration $\mathbf{G} := (\mathscr{G}_t)_{t\in\mathbb{R}_+}$ with

$$\mathscr{G}_t := \sigma\left((V_s, U_s) : s \le t\right), \quad t \in \mathbb{R}_+,$$

such that $U_0 = 0$ and $\exp(U)$ is a $\mathbf{G}$-martingale.

(v) L and (V,U) are independent.

We define the corresponding *logarithmic stock price process* by

$$X := \log(S_0) + L_V + U \tag{5.4}$$

and the corresponding *stock price process* by

$$S := \exp(X). \tag{5.5}$$

Definition 5.2.1 is quite abstract since we strive to conduct our analysis in a framework that is as general as possible. We demonstrate in Section 5.3 below that our setup encompasses a large class of stock price models from the literature.

The specification of the components of a stochastic volatility model implies that the corresponding stock price process is a martingale.

Theorem 5.2.2. *Let (S_0, L, V, U) be a stochastic volatility model. Then,* $\exp\left(\log(S_0) + L_V + U\right)$ *is a martingale with respect to the filtration generated by (L_V, V, U).*

PROOF. Cf. Section 5.7.1. □

From now on, we consider a fixed stochastic volatility model (S_0, L, V, U) that generates the stock price process S of interest.

We further consider a fixed European contingent claim on S with payoff $f(S_T)$ at maturity $T > 0$ for a measurable payoff function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $\mathrm{E}\left(|f(S_T)|\right) < \infty$. The further Assumption 5.2.21 will be imposed on f below. Working with the filtration generated by S, the first fundamental theorem of asset pricing [DS94] implies that a reasonable price for the option – in the sense that it is compatible with no-arbitrage – is given by

$$c := \mathrm{E}\left(f(S_T)\right) \tag{5.6}$$

since S is a martingale relative to P. In the sequel, we will be concerned with the derivation of reasonable approximations to this expectation.

Remark 5.2.3. 1. We do not discuss the precise definition of "no-arbitrage" here but refer to [DS94] for more details.

2. Expectation (5.6) leads also to a no-arbitrage price in the sense of [DS94] if S is a local martingale relative to P. However, we adhere to the common approach to work with a true martingale.

3. Typically, S allows for infinitely many different (local) martingale measures. We assume that the appropriate choice P has already been made, and we work with S relative to P.

5.2.2. Stochastic volatility model as perturbed Black-Scholes model

We derive our approximation to the option price c from (5.6) by employing our perturbation approach from Chapter 2. This requires the construction of a one-parametric curve in the space of possible stock price processes that connects the price process of interest S with a (time-dependent) Black-Scholes model.

A proposition for such a curve is made in this section. To this end, we separately perturb the components L, V, and U of the stochastic volatility model (S_0, L, V, U) that generates the stock price process S.

5.2.2.1. L as perturbed Brownian motion

Very similar to the curve proposed in Chapter 4 on approximate hedging in geometric Lévy models, we connect the Lévy process L with Brownian motion by a suitable time change, rescaling, and shift.

We set

$$D := \left\{ z \in \mathbb{C} : \mathrm{E}\left(e^{\mathrm{Re}(z)L_1}\right) < \infty \right\}. \tag{5.7}$$

By $\kappa : D \to \mathbb{C}$ we denote the cumulant generating function of L, i.e., the unique continuous function satisfying $\mathrm{E}\left(e^{zL_t}\right) = e^{\kappa(z)t}$ for all $t \in \mathbb{R}_+$ and $z \in D$. For existence and uniqueness of κ, cf. [Sat99, Lemma 7.6].

Note that $[0,1] + i\mathbb{R} \subset D$ since $\mathrm{E}\left(e^{L_1}\right) < \infty$ by assumption. This allows to define the family of processes L^λ, $\lambda \in (0,1]$, by

$$L_t^\lambda := -\frac{1}{\lambda^2} \log\left(\mathrm{E}\left(e^{\lambda L_1}\right)\right) t + \lambda L_{\frac{t}{\lambda^2}}, \quad t \in \mathbb{R}_+. \tag{5.8}$$

Assumption 5.2.4. We assume that L_1 has moments of any order.

Remark 5.2.5. For our derivation of second-order approximations, we need only the first five moments of L_1 to exist. However, we formulate the above and all following assumptions such that in principle approximations to any order can be computed. Besides this interesting mathematical aspect, this allows for easier exposition of the proofs. Moreover, since we require the first exponential moment of L_1 to exist, Assumption 5.2.4 is typically no restriction.

We obtain the following properties of L^λ.

Lemma 5.2.6. *For the family of processes L^λ, $\lambda \in (0,1]$, from* (5.8)*, we have the following:*

(i) For all $\lambda \in (0,1]$, L^λ is an intrinsic Lévy process with càdlàg paths independent of (V,U) such that $\mathrm{E}\left(e^{L_1^\lambda}\right) = 1$ *and* $\mathrm{Var}\left(L_1^\lambda\right) > 0$.

(ii) For all $\lambda \in (0,1]$ and $t \in \mathbb{R}_+$, we have $\mathrm{Var}\left(L_t^\lambda\right) = \mathrm{Var}(L_t) = t\,\mathrm{Var}(L_1)$.

(iii) For all $\lambda \in (0,1]$, we have $D^\lambda \subset D$, where

$$D^\lambda := \left\{ z \in \mathbb{C} : \mathrm{E}\left(e^{\mathrm{Re}(z) L_1^\lambda} \right) < \infty \right\}.$$

(iv) For all $\lambda \in (0,1]$, the cumulant generating function $\kappa^\lambda : D^\lambda \to \mathbb{C}$ of L^λ is given by

$$\kappa^\lambda(z) = -\frac{1}{\lambda^2}\kappa(\lambda)z + \frac{1}{\lambda^2}\kappa(\lambda z).$$

PROOF. Cf. Section 5.7.1. □

(5.8) does not make sense for $\lambda = 0$, but we obtain Brownian motion in the limit:

Lemma 5.2.7. *For $\lambda \to 0$, the family of Lévy processes $(L^\lambda)_{\lambda \in (0,1]}$ converges in law with respect to the Skorokhod topology (cf. [JS03, Section VI.1] for more details) to a Brownian motion:*

$$L^\lambda \xrightarrow{\mathscr{D}} -\frac{1}{2}\mathrm{Var}(L_1)I + \sqrt{\mathrm{Var}(L_1)}W \quad \text{as} \quad \lambda \to 0,$$

where I denotes the identity process $I_t = t$, and W is a standard Brownian motion.

PROOF. Cf. Section 5.7.1. □

We denote the limiting process by L^0, i.e.,

$$L_t^0 := -\frac{1}{2}\mathrm{Var}(L_1)t + \sqrt{\mathrm{Var}(L_1)}W_t, \quad t \in \mathbb{R}_+, \tag{5.9}$$

where the standard Brownian motion W shall live w.l.o.g. on the original probability space $(\Omega, \mathscr{F}, P)$.

The cumulant generating function κ^0 of L^0 is given by

$$\kappa^0(z) = \frac{1}{2}\mathrm{Var}(L_1)z(z-1), \quad z \in D^0 := \mathbb{C}. \tag{5.10}$$

5.2.2.2. U as perturbed 0

The component U of the stochastic volatility model of interest is interpreted as a perturbation of 0, and the corresponding curve connecting U with 0 is established by simple rescaling. However, it is our aim to obtain a stock price process that is generated by a stochastic volatility model for every point on the curve that connects S with geometric Brownian motion. In particular, this requires that the curve connecting U and 0 leads to a family of exponential martingales, which we assure by appropriate compensation.

Since U is a special semimartingale on the filtered probability space $(\Omega, \mathscr{F}, \mathbf{G}, P)$, we may represent it uniquely as

$$U = M + A,$$

where $M \in \mathscr{M}_{\mathrm{loc}}$ and $A \in \mathscr{V}$ and predictable.

Assumption 5.2.8. We assume that M is a martingale.

Assumption 5.2.9. We assume that M, as process on the filtered probability space $(\Omega, \mathscr{F}, \mathbf{G}, P)$, allows for differential characteristics (b^M, c^M, F^M) in the sense of Definition E.0.4 with respect to some truncation function $h^M : \mathbb{R} \to \mathbb{R}$.

For the construction of the curve for U, we rescale its martingale part M. We can represent the required exponential compensator of λM in terms of the differential characteristics of M.

Lemma 5.2.10. *For all $\lambda \in [0,1]$, λM as process on the filtered probability space $(\Omega, \mathscr{F}, \mathbf{G}, P)$ is exponentially special in the sense of Definition E.0.10. Its exponential compensator in the sense of Definition E.0.12 is given by the predictable process $K(\lambda M) \in \mathscr{V}$ with*

$$K(\lambda M)_t = \int_0^t \left(\lambda b_s^M + \frac{1}{2}\lambda^2 c_s^M + \int \left(e^{\lambda x} - 1 - \lambda h^M(x) \right) F_s^M(\mathrm{d}x) \right) \mathrm{d}s, \quad t \in \mathbb{R}_+, \tag{5.11}$$

i.e., $\exp(\lambda M - K(\lambda M)) \in \mathscr{M}_{\mathrm{loc}}$.

PROOF. Cf. Section 5.7.1. □

For $\lambda \in [0,1]$, we set

$$U^\lambda := \lambda M - K(\lambda M). \tag{5.12}$$

By the above Lemma 5.2.10, the process $\exp(U^\lambda)$ is a local martingale for all $\lambda \in [0,1]$. For $\lambda = 1$, it is by assumption a martingale. However, it is not obvious in general that this property transfers to the case $\lambda < 1$. Hence, we impose this as an assumption.

Assumption 5.2.11. We assume that for all $\lambda \in [0,1]$, the process $\exp(U^\lambda)$ is a martingale.

Since this assumption is often easy to check for concrete model specifications, we formulate it at this abstract level. The following proposition provides a sufficient condition for Assumption 5.2.11 to hold that only depends on M and not on the whole curve associated to U.

Proposition 5.2.12. *If* $\mathrm{E}\left(e^{M_t}\right) < \infty$ *for all* $t \in \mathbb{R}_+$, *then* $\exp(U^\lambda)$ *is a martingale for all* $\lambda \in [0,1]$.

PROOF. Cf. Section 5.7.1. □

Remark 5.2.13. If we want to check for the stochastic volatility model of interest (S_0, L, V, U) that $\exp(U)$ is a martingale and not only a local martingale, it is sufficient that there exists $\varepsilon > 0$ such that $\mathrm{E}\left(e^{(1+\varepsilon)M_t}\right) < \infty$ for all $t \in \mathbb{R}_+$. (The proof is the same as for Proposition 5.2.12.)

Remark 5.2.14. Many stock price models of practical importance (cf. Section 5.3) are generated by a stochastic volatility model in the sense of Definition 5.2.1 such that in particular M is the component of an affine process in the sense of [Kal06]. This implies that U^λ is affine as well. Sufficient conditions when an exponentially affine local martingale is a martingale are provided by [KMK10]. In particular, continuity of the process is a sufficient condition.

5.2.2.3. V as perturbed deterministic function

The remaining component V of the stochastic volatility model (S_0, L, V, U) of interest is interpreted as a perturbed deterministic function, for which $t \mapsto \mathrm{E}(V_t)$ is a canonical candidate. In addition, our specification involves the variance process of M. Let us give our definition of V^λ before explaining this.

Assumption 5.2.15. We assume that for all $t \in \mathbb{R}_+$, the random variables V_t and M_t have moments of any order.

Assumption 5.2.16. We assume that M is quasi-left-continuous in the sense of [JS03, Definition I.2.25].

Remark 5.2.17. In particular, continuous processes are quasi-left-continuous. Moreover, every Lévy process is quasi-left-continuous, cf. [JS03, Theorem II.4.15].

Lemma 5.2.18. *(i) $t \mapsto \mathrm{E}(V_t)$ is real-valued, continuous, and strictly increasing.*

(ii) $t \mapsto \mathrm{Var}(M_t)$ is real-valued, continuous, and increasing.

PROOF. Cf. Section 5.7.1. □

For $\lambda \in [0,1]$, we define the process V^λ by

$$V_t^\lambda := \lambda V_t + (1-\lambda)\mathrm{E}(V_t) + \left(1-\lambda^2\right) \frac{\mathrm{Var}(M_t)}{\mathrm{Var}(L_1)}, \quad t \in \mathbb{R}_+. \tag{5.13}$$

Having specified L^λ, U^λ, V^λ, $\lambda \in [0,1]$, we will consider the family of stock price processes generated by the family of stochastic volatility models $(S_0, L^\lambda, V^\lambda, U^\lambda)$. By construction, $\mathrm{Var}\left(U^\lambda\right) \to 0$ as $\lambda \to 0$. To account for this "lacking variance", the related variance process of M is incorporated into the definition of V^λ.

5.2.2.4. Summary of the perturbation

Proposition 5.2.19. *For all $\lambda \in [0,1]$, $(S_0, L^\lambda, U^\lambda, V^\lambda)$ is a stochastic volatility model in the sense of Definition 5.2.1.*

PROOF. The required properties in Definition 5.2.1 follow from Lemma 5.2.6(i), Assumption 5.2.11, and Lemma 5.2.18. (The independence of L^λ and (V^λ, U^λ) for all $\lambda \in [0,1]$ is clear by construction.) □

This gives rise to the family of (logarithmic) stock price processes generated by $(S_0, L^\lambda, U^\lambda, V^\lambda)$: for $\lambda \in [0,1]$, we set

$$X^\lambda := \log(S_0) + L^\lambda_{V^\lambda} + U^\lambda, \tag{5.14}$$

$$S^\lambda := \exp(X^\lambda). \tag{5.15}$$

Let us summarize the curve we have established in the following

Proposition 5.2.20. *The family of stock price processes S^λ, $\lambda \in [0,1]$, defined in* (5.15) *satisfies the following:*

(i) $S^1 = S$.

(ii) The random variable S^0_T coincides in law with the discounted risk-neutral stock price at time T in a Black-Scholes model with volatility

$$\overline{\sigma} := \sqrt{\frac{1}{T}\left(\mathrm{E}\left(V_T\right)\mathrm{Var}\left(L_1\right) + \mathrm{Var}\left(M_T\right)\right)} \tag{5.16}$$

and initial stock price S_0, i.e., $S^0_T \stackrel{\mathscr{D}}{=} S_0 \exp\left(-\frac{1}{2}\overline{\sigma}^2 T + \overline{\sigma} Z\right)$ for a $N(0,T)$-distributed random variable Z. If the derivative

$$\sigma(t) := \sqrt{\frac{\partial}{\partial t}\left(\mathrm{Var}\left(L_1\right)\mathrm{E}\left(V_t\right) + \mathrm{Var}\left(M_t\right)\right)}, \quad t \in \mathbb{R}_+, \tag{5.17}$$

exists and $\lim_{t\to\infty} \int_0^t \sigma^2(s)\,\mathrm{d}s = \infty$, *then*

$$S^0_t = S_0 \exp\left(-\frac{1}{2}\int_0^t \sigma^2(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\overline{W}_s\right), \quad t \in \mathbb{R}_+,$$

for a standard Brownian motion $\overline{W}$, i.e., S^0 is the discounted risk-neutral stock price process in a Black-Scholes model with time-dependent volatility function σ and initial stock price S_0. Then in particular, $\int_0^T \sigma^2(t)\,\mathrm{d}t = \overline{\sigma}^2 T$.

PROOF. Cf. Section 5.7.1. □

5.2.3. Further assumptions

In order to carry out our analysis on option prices, we need to impose a number of additional assumptions that we state and comment in this section.

We denote by (b^L, c^L, F^L) the Lévy-Khintchine triplet resp. the differential characteristics (cf. Remark E.0.5) of L relative to some truncation function $h^L : \mathbb{R} \to \mathbb{R}$.

Assumption 5.2.21. We assume that there exists $R \in \mathbb{R}$ such that the following holds.

1. (Regularity of L and V or smoothness of f)

   a) There exists $p : (R + i\mathbb{R}) \to \mathbb{C}$ with $x \mapsto |p(R+ix)|$ being integrable such that f admits the representation

   $$f(s) = \int_{R-i\infty}^{R+i\infty} s^z p(z)\,\mathrm{d}z, \quad s \in \mathbb{R}_+, \tag{5.18}$$

   and there are $b_1, b_2, \beta > 0$ such that

   $$\mathrm{E}\left(e^{uV_T}\right) \le e^{b_1 - b_2|u|^\beta} \quad \text{for all } u \in \mathbb{R}_-, \tag{5.19}$$

and, if $c^L = 0$, there is $0 < \gamma < 2$ such that

$$\liminf_{r \to 0} r^{\gamma - 2} \int_{-r}^{r} x^2 F^L(\mathrm{d}x) > 0. \tag{5.20}$$

b) Alternatively to (1a), we can assume that f is in $C^\infty(\mathbb{R}_+, \mathbb{R})$ and all derivatives of $x \mapsto f(e^x)e^{-Rx}$ are integrable on $\mathbb{R}$.

2. $R \in \operatorname{int} D$.
3. $\mathrm{E}\left(\exp\left(\left(\max_{\lambda \in [0,1]} 4\lambda\, \kappa^\lambda(R)\right) V_T\right)\right) < \infty$.
4. $\mathrm{E}\left(e^{2RM_T}\right) < \infty$ if $R > 0$, and $\mathrm{E}\left(e^{2RU_T}\right) < \infty$ if $R < 0$.

Remark 5.2.22. 1. Assumption 5.2.21(1a) demands in particular the integral representation (5.18) for the payoff function f, which will be a key point for our analysis. How such a representation is obtained and how it is related to option prices is discussed in detail in Chapter 3.

2. (5.19) means that the Laplace transform of V_T decays exponentially fast. In many models of practical importance, V_T is the integral over a mean-reverting process, which makes it intuitively plausible that (5.19) typically holds.

3. (5.20) implies that a decay condition similar to (5.19) holds for the moment generating function of L_1^λ uniformly for $\lambda \in [0,1]$, cf. Lemma 5.7.5 below. Since the Lévy measure F^L is typically known for parametric models of practical relevance, (5.20) can be checked in concrete specifications. It is satisfied, e.g., for the normal inverse Gaussian Lévy process [BN98], the CGMY Lévy process [CGMY02], or the generalized hyperbolic Lévy process [EP00]. It does not hold for the variance gamma Lévy process [MCC98].

4. (5.19) and (5.20) are assumed to ensure that approximations to option prices can be computed up to arbitrary order. If one restricts oneself to lower order approximations, the conditions can be weakened in the sense that the Laplace transform and the moment generating function of V_T and L_1^λ need to exhibit power decay of a certain order. For ease of exposition, we work with the exponential decay conditions.

5. If (5.19) or (5.20) do not hold, we can alternatively work with a smooth payoff function in the sense of 5.2.21(1b). This implies in particular Representation (5.18), cf. Section 3.3 in Chapter 3.

6. Given that we work with integral representations of the prices of the options with payoff function f in the models S^λ, 5.2.21(2) and 5.2.21(4) are quite natural. The factor 2 in the exponential of the latter conditions is required to apply Cauchy-Schwartz arguments to separate complicated expressions. For similar reasons, we need 5.2.21(3).

From now on, we fix some $R \in \mathbb{R}$ satisfying Assumption 5.2.21.

Condition 5.2.21(3) involves the perturbation parameter $\lambda \in [0,1]$ and is unhandy to check. The following proposition allows to verify 5.2.21(3) only in dependence on the Lévy-Khintchine triplet of L.

Proposition 5.2.23. *(i) For all $A \in [0,1]$, we have that $A \in D^\lambda$ and $\kappa^\lambda(A) \leq 0$ for all $\lambda \in [0,1]$.*

(ii) For all $A \in [0,1]^c$ such that $A \in D$, we have $A \in D^\lambda$ for all $\lambda \in [0,1]$, and

$$\max_{\lambda\in[0,1]} 4\lambda\,\kappa^\lambda(A) \leq 2c^L(A^2 - A) + 2\int \left(A^2x^2(e^{Ax} \vee 1) + |A|\,1_{\{A<0\}}x^2(e^x \vee 1)\right) F^L(\mathrm{d}x). \tag{5.21}$$

PROOF. Cf. Section 5.7.1. □

Remark 5.2.24. 1. If R can be chosen in $[0,1]$, Proposition 5.2.23(i) implies that Assumption 5.2.21(3) is automatically satisfied since $V_T \geq 0$.

2. Otherwise, the upper bound for $\max_{\lambda\in[0,1]} 4\lambda\,\kappa^\lambda(R)$ provided in Proposition 5.2.23(ii) can be computed at least numerically if the Lévy density F^L of L is known, which is often the case.

3. Note that if L is Brownian motion, then by construction $\kappa^\lambda(R) = \kappa^1(R)$ for all $\lambda \in [0,1]$, i.e., the maximum is attained for $\lambda = 1$ in the case $R \in [0,1]^c$.

The following assumption is typically not restrictive and also designed to allow for arbitrarily high orders of approximation.

Assumption 5.2.25. We assume that the random variable $\langle M, M\rangle_T$ has moments of any order.

The following assumption is made to admit our analysis in full generality. In all model classes presented in Section 5.3, we have $F_s^M \equiv 0$, and hence the assumption is trivially satisfied in these important cases.

Assumption 5.2.26. We assume that $\int_0^T \int (e^x \vee 1)\,|x|^n\, F_s^M(\mathrm{d}x)\,\mathrm{d}s$ exists and has moments of any order for all $n \in \mathbb{N}_{\geq 2}$.

An overview of all assumptions required for our analysis is provided in Table 5.1.

5.2.4. Quantity to approximate

The goal is to provide an approximation to the initial option price $c = \mathrm{E}(f(S_T))$ relative to the stock price process of interest S. In order to employ our perturbation approach from Chapter 2 using the curve S^λ, $\lambda \in [0,1]$, we must guarantee that the corresponding option prices exist relative to all stock price processes S^λ, $\lambda \in [0,1]$, and that the dependence on λ is smooth enough.

Proposition 5.2.27. *For all $\lambda \in [0,1]$, we have $\mathrm{E}\left(\left|f(S_T^\lambda)\right|\right) < \infty$. Setting*

$$c^\lambda := \mathrm{E}\left(f(S_T^\lambda)\right), \tag{5.22}$$

we have that $\lambda \mapsto c^\lambda$ is in $C^\infty([0,1], \mathbb{R}_+)$.

PROOF. Cf. Propositions 5.7.11 and 5.7.15 in Section 5.7.3. □

Assumption	**Mathematical condition**
5.2.4	L_1 has moments of any order.
5.2.8	M is a martingale.
5.2.9	M allows for differential characteristics.
5.2.11	e^{U^λ} is a martingale for all $\lambda \in [0,1]$.
5.2.15	For all $t \in \mathbb{R}_+$, V_t and M_t have moments of any order.
5.2.16	M is quasi-left-continuous.
5.2.21	There exists $R \in \mathbb{R}$ such that 5.2.21(1–4) hold.
5.2.21(1)	The payoff function f allows for an integral representation along the line with real part R, and there exist b_1, b_2, $\beta > 0$ such that $$\mathrm{E}\left(e^{uV_T}\right) \leq e^{b_1 - b_2\lvert u\rvert^\beta} \quad \text{for all } u \in \mathbb{R}_-,$$ and, if L has no Brownian component, there is $0 < \gamma < 2$ such that $$\liminf_{r\to 0} r^{\gamma-2} \int_{-r}^{r} x^2\, F^L(\mathrm{d}x) > 0.$$ Alternatively to these conditions: f is smooth and all derivatives of $x \mapsto f(e^x)e^{-Rx}$ are integrable on $\mathbb{R}$.
5.2.21(2)	There is $\varepsilon > 0$ such that $\mathrm{E}\left(e^{(R\pm\varepsilon)L_1}\right) < \infty$.
5.2.21(3)	$\mathrm{E}\left(\exp\left(\left(\max_{\lambda\in[0,1]} 4\lambda\, \kappa^\lambda(R)\right) V_T\right)\right) < \infty$.
5.2.21(4)	$\mathrm{E}\left(e^{2RM_T}\right) < \infty$ if $R > 0$, and $\mathrm{E}\left(e^{2RU_T}\right) < \infty$ if $R < 0$.
5.2.25	$\langle M, M\rangle_T$ has moments of any order.
5.2.26	$\int_0^T \int (e^x \vee 1)\, \lvert x\rvert^n\, F_s^M(\mathrm{d}x)\, \mathrm{d}s$ has moments of any order for all $n \in \mathbb{N}_{\geq 2}$.

Table 5.1.: Overview of assumptions on stochastic volatility model

We can now rephrase the abstract Principle 2.2.1 of second-order approximation in our specific context.

Definition 5.2.28. We call

$$\mathfrak{A}(c) := \mathfrak{A}_0(c) + \mathfrak{A}_1(c) + \frac{1}{2}\mathfrak{A}_2(c)$$

with

$$\mathfrak{A}_0(c) = c^0, \quad \mathfrak{A}_1(c) = \left.\frac{\partial}{\partial\lambda}c^\lambda\right|_{\lambda=0}, \quad \left.\frac{\partial^2}{\partial\lambda^2}c^\lambda\right|_{\lambda=0}$$

with c^λ from (5.22) *second-order approximation* to the initial option price c (relative to the curve S^λ, $\lambda \in [0,1]$).

5.3. Models from the literature within our framework

In this section, we demonstrate that our framework encompasses several classes of stock price models from the literature. We reuse some variables (like S, M, or W) introduced in Section 5.2 in order to avoid unintuitive notation. Moreover, we do not go into mathematical details but aim at an easily accountable overview.

5.3.1. Geometric Lévy models

In geometric Lévy models, the discounted price process of the stock is given by

$$S_t = S_0 e^{L_t}, \quad t \in \mathbb{R}_+,$$

for the initial stock price $S_0 > 0$ and a Lévy process L. Important parametric classes for L include the normal inverse Gaussian [BN95], the variance gamma [MS90], or the CGMY [CGMY02] Lévy process. Textbooks treating this class of models in great detail are [Sch03, CT03]. From the statistical point of view, geometric Lévy processes allow for asymmetry and heavy tails of returns and thus capture some important empirical facts of asset returns in a better way than the Black-Scholes model. From the risk-neutral perspective, the smile or skew of implied volatilities on real markets can be covered at least for fixed maturities.

Obviously, S is generated by the stochastic volatility model $(S_0, L, I, 0)$ in the sense of Definition 5.2.1, where I denotes the identity process.

5.3.2. Bivariate diffusion models

In typical bivariate diffusion models, the logarithm of the discounted stock price process relative to the risk-neutral measure follows the stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}g(y_t)^2\,\mathrm{d}t + g(y_t)\,\mathrm{d}W_t^1, \quad X_0 = \log(S_0), \\ \mathrm{d}y_t &= \mu_y(y_t)\,\mathrm{d}t + \sigma_y(y_t)\,\mathrm{d}W_t^2, \quad y_0 > 0, \end{aligned} \tag{5.23}$$

for the initial stock price $S_0 > 0$, functions g, μ_y, σ_y and standard Brownian motions W^1, W^2 with $\mathrm{d}\langle W^1, W^2\rangle_t = \rho\,\mathrm{d}t$ for $\rho \in [-1,1]$. The process $g(y)$ is often called *instantaneous volatility process*. Equivalently, (5.23) can be written as

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}g(y_t)^2\,\mathrm{d}t + g(y_t)\left(\sqrt{1-\rho^2}\,\mathrm{d}Z_t^1 + \rho\,\mathrm{d}Z_t^2\right), \quad X_0 = \log(S_0), \\ \mathrm{d}y_t &= \mu_y(y_t)\,\mathrm{d}t + \sigma_y(y_t)\,\mathrm{d}Z_t^2, \quad y_0 > 0, \end{aligned} \tag{5.24}$$

for two independent standard Brownian motions Z^1, Z^2. Prominent examples are the Heston model [Hes93], where $g(x) = \sqrt{x}$, $\mu_y(x) = \kappa(\eta - x)$, $\sigma_y(x) = \theta\sqrt{x}$ for constants $\kappa, \eta, \theta > 0$, or the (generalized) Stein & Stein model [SS91, SZ99], where $g(x) = x$, $\mu_y(x) = \kappa(\eta - x)$, $\sigma_y(x) = \theta$ for constants $\kappa, \eta, \theta > 0$. We consider these examples both in our numerical study in Section 5.5 below.

The key to embed bivariate diffusion models into our framework is provided by Proposition 5.3.2 below, which is based on the time change representation of Brownian integrals given by the famous

Theorem 5.3.1 (Dambis, Dubins-Schwarz)**.** *On a probability space equipped with a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, consider a continuous local martingale M with $M_0 = 0$ and such that $\langle M, M\rangle_\infty = \infty$. Moreover, set $T_t := \inf\{s \in \mathbb{R}_+ : \langle M, M\rangle_s > t\}$. Then, the process B defined by $B_t := M_{T_t}$ is a standard Brownian motion with respect to the filtration $(\mathscr{F}_{T_t})_{t\in\mathbb{R}_+}$, and $M = B_{\langle M,M\rangle}$.*

PROOF. Cf., e.g., [RY91, Theorem V.1.6]. □

Proposition 5.3.2. *On a filtered probability space, consider a standard Brownian motion W and a process $y \in L(W)$ such that $\int_0^\infty y_t^2\,\mathrm{d}t = \infty$. Then, there exists a standard Brownian motion $\widetilde{W}$ (in general relative to a different filtration) such that for all $t \in \mathbb{R}_+$*

$$\int_0^t y_s\,\mathrm{d}W_s = \widetilde{W}_{\int_0^t y_s^2\,\mathrm{d}s}. \tag{5.25}$$

If W and y are independent, then $\widetilde{W}$ can be chosen such that it is independent of y as well.

PROOF. Cf. Section 5.7.2. □

Remark 5.3.3. The time change representation (5.25) of the stochastic integral does not transfer to general integrators. At the bottom of Relation (5.25) lies the self-similarity of Brownian motion. The equation therefore continues to hold if Brownian motion is replaced by a symmetric α-stable Lévy process, but not beyond, cf. [KS02b].

In light of Proposition 5.3.2 – given the necessary regularity of y – the stock price process $S = e^X$ in a bivariate diffusion model as in (5.24) is generated by the stochastic volatility model (S_0, L, V, U) in the sense of Definition 5.2.1 with

$$\begin{aligned} L_t &= -\frac{1}{2}\left(1-\rho^2\right)t + \sqrt{1-\rho^2}B_t, \\ V_t &= \int_0^t g(y_s)^2 \, \mathrm{d}s, \\ U_t &= -\frac{1}{2}\rho^2 \int_0^t g(y_s)^2 \, \mathrm{d}s + \rho \int_0^t g(y_s) \, \mathrm{d}Z_s^2 \end{aligned}$$

for a suitable standard Brownian motion B independent of Z^2 and y as in (5.24).

5.3.3. Models according to Barndorff-Nielsen & Shephard

In the class of models introduced by Barndorff-Nielsen, Shephard, and co-authors (cf. [BNS01, BNNS02, BNS03]), the logarithm of the discounted stock price process is given by

$$\begin{aligned} \mathrm{d}X_t &= -\left(\kappa m(\rho) + \frac{1}{2}y_t^2\right)\mathrm{d}t + \sqrt{y_t}\,\mathrm{d}W_t + \rho\,\mathrm{d}Z_{\kappa t}, \quad X_0 = \log(S_0), \\ \mathrm{d}y_t &= -\kappa y_t\,\mathrm{d}t + \mathrm{d}Z_{\kappa t}, \quad y_0 > 0, \end{aligned} \tag{5.26}$$

for constants $\kappa > 0$, $\rho \leq 0$, a standard Brownian motion W, an increasing Lévy process Z (i.e., a subordinator) independent of W, and $m(\rho) := \log\left(\mathrm{E}\left(e^{\rho Z_1}\right)\right)$. In contrast to the class of bivariate diffusion models from Section 5.3.2, the process y driving instantaneous volatility is not continuous but has jumps. Common choices for Z in applications are a gamma or inverse Gaussian process, cf. also [Sch03, Section 7.1] for further details.

Using Proposition 5.3.2, we see that the corresponding stock price process $S = e^X$ is generated by the stochastic volatility model (S_0, L, V, U) in the sense of Definition 5.2.1 with

$$\begin{aligned} L_t &= -\frac{1}{2}t + B_t, \\ V_t &= \int_0^t y_s \, \mathrm{d}s, \\ U_t &= -\kappa m(\rho)t + \rho Z_{\kappa t} \end{aligned}$$

for a standard Brownian motion B independent of y.

5.3.4. Time-changed Lévy models according to Carr et. al (2003)

[CGMY03] consider models where the logarithmic stock price process is given by a Lévy process subordinated by a suitable independent stochastic process representing the random clock of market activity resp. stochastic volatility. The authors also allow for correlation between movements in the stock price and stochastic volatility. Hence, this class of models combines the ideas from

Sections 5.3.1–5.3.3 above. More specifically, [CGMY03] model the logarithm of the risk-neutral discounted stock price process by

$$X_t = \log(S_0) + L_{\int_0^t y_s \,\mathrm{d}s}, \quad t \in \mathbb{R}_+,$$

where $S_0 > 0$, L is a normal inverse Gaussian, variance gamma, or CGMY Lévy process such that $\exp(L)$ is an intrinsic martingale. Moreover, y follows either a square root process, i.e.,

$$\mathrm{d}y_t = \kappa(\eta - y_t)\,\mathrm{d}t + \theta\sqrt{y_t}\,\mathrm{d}W_t, \quad y_0 > 0,$$

for constants κ, η, $\theta > 0$ and a standard Brownian motion W independent of L, or a Lévy-driven Ornstein-Uhlenbeck process, i.e.,

$$\mathrm{d}y_t = -\kappa y_t\,\mathrm{d}t + \mathrm{d}Z_t, \quad y_0 > 0,$$

for a constant $\kappa > 0$ and an increasing Lévy process Z independent of L. The authors explicitly consider the cases where Z is a gamma or inverse Gaussian process, but other choices for L and Z are possible as well. In order to allow for a relation between changes in the stock price and stochastic volatility, [CGMY03] consider the extensions

$$X_t = \log(S_0) + L_{\int_0^t y_s \,\mathrm{d}s} + \rho\theta\int_0^t \sqrt{y_s}\,\mathrm{d}W_s - \frac{1}{2}\rho^2\theta^2\int_0^t y_s\,\mathrm{d}s, \quad t \in \mathbb{R}_+, \tag{5.27}$$

in the case that y is given by a square root process resp.

$$X_t = \log(S_0) + L_{\int_0^t y_s \,\mathrm{d}s} + \rho Z_t - m(\rho)t, \quad t \in \mathbb{R}_+, \tag{5.28}$$

if y is given by an Ornstein-Uhlenbeck process, where $m(\rho) := \log\left(\mathrm{E}\left(e^{\rho Z_1}\right)\right)$, $\rho \in \mathbb{R}$. Hence, the related discounted stock price process $S = e^X$ is generated by the stochastic volatility model (S_0, L, V, U) in the sense of Definition 5.2.1 with

$$\begin{aligned} V_t &= \int_0^t y_s\,\mathrm{d}s, \\ U_t &= \rho y_t - \frac{1}{2}\rho^2\theta^2 V_t \end{aligned}$$

if y is given by a square root process resp.

$$\begin{aligned} V_t &= \int_0^t y_s\,\mathrm{d}s, \\ U_t &= \rho Z_t - m(\rho)t \end{aligned}$$

if y is given by a Lévy-driven Ornstein-Uhlenbeck process. We finally mention that [CGMY03] consider also similar stock price models but such that the discounted stock price process exhibits constant expectation and is not necessarily a martingale. The authors relate this to the absence of certain static arbitrage opportunities. We do not go into detail on this topic but refer to [CGMY03, Section 5].

5.4. Approximation to the option price

In this section, we present our second-order approximation to the initial option price c from (5.6) in the sense of Definition 5.2.28.

5.4.1. Components of the approximation

The stochastic volatility model (S_0, L, V, U) enters into the approximation formula via moments of L, V, and the martingale part M of U. The payoff function f emerges via corresponding option price sensitivities in the limiting Black-Scholes model S^0.

5.4.1.1. Moments of L, V and M

For the first four moments of the Lévy process L, we obtain for $t \in \mathbb{R}_+$

$$\mathrm{E}(L_t) = \mathrm{E}(L_1)\,t, \qquad \mathrm{Var}(X_t) = \mathrm{Var}(L_1)\,t,$$

$$\mathrm{Skew}(L_t) = \mathrm{Skew}(L_1)\,\frac{1}{\sqrt{t}}, \qquad \mathrm{ExKurt}(L_t) = \mathrm{ExKurt}(L_1)\,\frac{1}{t}.$$

Here, Skew(Y) and ExKurt(Y) denote skewness and excess kurtosis of a random variable Y, i.e.,

$$\mathrm{Skew}(Y) := \frac{\mathrm{E}\left((Y-\mathrm{E}(Y))^3\right)}{\sqrt{\mathrm{Var}(Y)}^3} \quad \text{and} \quad \mathrm{ExKurt}(Y) := \frac{\mathrm{E}\left((Y-\mathrm{E}(Y))^4\right)}{\sqrt{\mathrm{Var}(Y)}^4} - 3$$

if $\mathrm{E}\left(Y^4\right) < \infty$. Due to the scaling property in time, we refer to $\mathrm{E}(L_1)$, $\sqrt{\mathrm{Var}(L_1)}$, $\mathrm{Skew}(X_1)$, and $\mathrm{ExKurt}(X_1)$ as *drift*, *volatility*, *skewness rate*, and *excess kurtosis rate* of the Lévy process L.

In addition, we require the second-order moment structure of (V_T, M_T), or more specifically $\mathrm{E}(V_T)$, $\mathrm{Var}(V_T)$, $\mathrm{Var}(M_T)$, $\mathrm{Cov}(V_T, M_T)$. Note that $\mathrm{E}(M_T) = 0$ since M is a martingale by Assumption 5.2.8. We need to consider the moments of the process (V, M) explicitly at maturity T since a rescaling property as for L does not hold in general.

5.4.1.2. Cash greeks in the Black-Scholes model

By Proposition 5.2.20 above, S_T^0 is the discounted stock price at time T in a Black-Scholes model with volatility parameter

$$\overline{\sigma} = \sqrt{\frac{1}{T}(\mathrm{E}(V_T)\,\mathrm{Var}(L_1) + \mathrm{Var}(M_T))}.$$

Letting the initial price of the option with payoff $f(S_T^0)$ in this model depend on the initial stock price, we obtain the pricing function $C : \mathbb{R}_+ \to \mathbb{R}$ with

$$C(s) := \mathrm{E}\left(f\left(se^{-\frac{1}{2}\overline{\sigma}^2 T + \overline{\sigma} W_T}\right)\right) \tag{5.29}$$

for the standard Brownian motion W.

Lemma 5.4.1. *The function $C : \mathbb{R}_+ \to \mathbb{R}$, $s \mapsto C(s)$, from* (5.29) *is infinitely often differentiable.*

PROOF. Cf. Lemma 3.4.1. □

For $n \in \mathbb{N}$, the quantity $\frac{\partial^n}{\partial s^n} C(s)$ represents the n-th order sensitivity of the initial option price with respect to changes in the initial stock price. Such sensitivities are often referred to as *greeks*. Here, we consider so-called *cash greeks*, where the sensitivity is multiplied by the corresponding power of the stock price.

Definition 5.4.2. For $n \in \mathbb{N}$, set

$$D_n(s) := s^n \frac{\partial^n}{\partial s^n} C(s), \quad s \in \mathbb{R}_+,$$

with function $C : \mathbb{R}_+ \to \mathbb{R}$, $s \mapsto C(s)$, from (5.29).

5.4.2. Main result

The next theorem on the approximation of the option price is the core of this chapter.

Theorem 5.4.3 (Second-order approximation to the option price)**.** *The second-order approximation to the initial option price c from* (5.6) *in the sense of Definition 5.2.28 is given by*

$$\mathfrak{A}(c) = \mathfrak{A}_0(c) + \mathfrak{A}_1(c) + \frac{1}{2}\mathfrak{A}_2(c)$$

with

$$\begin{aligned}
\mathfrak{A}_0(c) &= C(S_0), \\
\mathfrak{A}_1(c) &= \frac{1}{6}\mathrm{Skew}(L_1)\sqrt{\mathrm{Var}(L_1)}\,\overline{\sigma}^2 T \sum_{k=1}^{3} a_k D_k(S_0), \\
\mathfrak{A}_2(c) &= \frac{1}{36}\mathrm{Skew}(L_1)^2\,\mathrm{Var}(L_1)\,\overline{\sigma}^4 T^2 \sum_{k=1}^{6} b_k D_k(S_0) \\
&\quad + \frac{1}{4}\mathrm{Var}(L_1)^2\,\mathrm{Var}(V_T) \sum_{k=1}^{4} d_k D_k(S_0) \\
&\quad + \mathrm{Var}(L_1)\,\mathrm{Cov}(V_T, M_T) \sum_{k=1}^{3} e_k D_k(S_0) \\
&\quad + \frac{1}{12}\mathrm{ExKurt}(L_1)\,\mathrm{Var}(L_1)\,\overline{\sigma}^2 T \sum_{k=1}^{4} g_k D_k(S_0),
\end{aligned}$$

where

$$\begin{aligned}
a &= (0,3,1,0,0,0)^\top, & b &= (0,18,78,63,15,1)^\top, \\
d &= (0,2,4,1,0,0)^\top, & e &= (0,2,1,0,0,0)^\top, \\
g &= (0,7,6,1,0,0)^\top.
\end{aligned}$$

Here, $C(S_0)$ is the Black-Scholes option price relative to volatility parameter

$$\overline{\sigma} = \sqrt{\frac{1}{T}\left(\mathrm{E}\left(V_T\right)\mathrm{Var}\left(L_1\right) + \mathrm{Var}\left(M_T\right)\right)}$$

from (5.29)*, and $D_k(S_0)$, $k = 1, \ldots, 6$, are the related cash greeks as in Definition 5.4.2.*

PROOF. Cf. Proposition 5.7.15. □

Remark 5.4.4. Like our approximations to quadratic hedging from Chapter 4, the second-order approximation to the option price disentangles the price process of the stock and the payoff function of the option. The price process enters via the lower-order moments of its components, and the payoff function manifests itself via the corresponding cash greeks in the limiting Black-Scholes model. For many parametric models of practical importance, the required moments can be computed as closed-form expressions in terms of the model parameters. We demonstrate this for four important models in Section 5.5. The Black-Scholes cash greeks are either known in closed form for simple options like calls and puts, cf. Appendix D, or they can be efficiently evaluated numerically via their integral representation from Lemma 3.4.3.

Remark 5.4.5. Let us consider the function

$$\widetilde{C}(\tau) := \mathrm{E}\left(f\left(S_0 e^{-\frac{1}{2}\tau T + \sqrt{\tau} W_T}\right)\right), \quad \tau > 0,$$

i.e., the Black-Scholes price of the option with initial stock price S_0, payoff function f, and maturity T in dependence on the *squared* volatility parameter. By inspection of the proof of Lemma 3.4.3, it is easy to see that

$$\frac{\partial^2}{\partial \tau^2}\widetilde{C}(\overline{\sigma}^2) = \frac{1}{4}T^2\left(D_4(S_0) + 4D_3(S_0) + 2D_2(S_0)\right),$$

which corresponds to the linear combination of cash greeks appearing in the summand related to $\mathrm{Var}\left(V_T\right)$ in our approximation. In a bivariate diffusion model with zero correlation (cf. Section 5.3.2), we have $\mathrm{Skew}\left(L_1\right) = \mathrm{ExKurt}\left(L_1\right) = 0$, $\mathrm{Var}\left(L_1\right) = 1$, $\mathrm{Cov}\left(V_T, M_T\right) = 0$. Hence, in this case our second-order approximation to the option price reduces to

$$\mathfrak{A}\left(c\right) = C(S_0) + \frac{1}{2}\mathrm{Var}\left(\frac{1}{T}V_T\right)\frac{\partial^2}{\partial \tau^2}\widetilde{C}(\overline{\sigma}^2),$$

which coincides with the second-order power series approximation in bivariate diffusion models according to [HW87] and [BR94], cf. Section 5.6.2.

5.4.3. Put-call parity

Interestingly, put-call parity continues to hold for our second-order approximation to the option price from Theorem 5.4.3.

Theorem 5.4.6 (Put-call parity for second-order approximation to the option price)**.** *Assume that we can apply our setup from Section 5.2 to the payoff functions of European call and put options with the same maturity $T > 0$ and discounted strike $K > 0$, i.e., $f_{\text{call}}(s) = (s-K)^+$ and $f_{\text{put}}(s) = (K-s)^+$. (Note that integral representations of these functions are given by Example 3.2.3.) Denoting the corresponding option prices relative so S by c_{call} and c_{put}, their second-order approximations $\mathfrak{A}\left(c_{\text{call}}\right)$ and $\mathfrak{A}\left(c_{\text{put}}\right)$ in the sense of Definition 5.2.28 satisfy the put-call parity, i.e.,*

$$\mathfrak{A}\left(c_{\text{call}}\right) - \mathfrak{A}\left(c_{\text{put}}\right) = S_0 - K.$$

PROOF. The put-call parity applied in the Black-Scholes model implies that all Black-Scholes cash greeks as in Definition 5.4.2 of order 2 and higher coincide for call and put options with the same strike and maturity, cf. also the explicit formulas in Appendix D. Hence,

$$\mathfrak{A}_1\left(c_{\text{call}}\right) = \mathfrak{A}_1\left(c_{\text{put}}\right), \quad \text{and} \quad \mathfrak{A}_2\left(c_{\text{call}}\right) = \mathfrak{A}_2\left(c_{\text{put}}\right),$$

which implies

$$\mathfrak{A}\left(c_{\text{call}}\right) - \mathfrak{A}\left(c_{\text{put}}\right) = \mathfrak{A}_0\left(c_{\text{call}}\right) - \mathfrak{A}_0\left(c_{\text{put}}\right).$$

However, $\mathfrak{A}_0\left(c_{\text{call}}\right)$ and $\mathfrak{A}_0\left(c_{\text{put}}\right)$ are the Black-Scholes prices of the call and put option with respect to the same volatility parameter, and as such they satisfy the put-call parity, i.e.,

$$\mathfrak{A}_0\left(c_{\text{call}}\right) - \mathfrak{A}_0\left(c_{\text{put}}\right) = S_0 - K,$$

which yields the assertion. □

5.4.4. Approximation to implied volatility

Let us consider the function $\widehat{C} : (0,\infty) \to \mathbb{R}$, $\sigma \mapsto \widehat{C}(\sigma)$, with

$$\widehat{C}(\sigma) := \mathrm{E}\left(f\left(S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma W_T}\right)\right), \tag{5.30}$$

i.e., the initial price of the option with discounted payoff function f and maturity T in a Black-Scholes model with initial price S_0, depending on the volatility parameter σ. Observe that we have $\widehat{C}(\overline{\sigma}) = C(S_0)$ for function C from (5.29).

Lemma 5.4.7. *The function $\widehat{C} : (0,\infty) \to \mathbb{R}$, $\sigma \mapsto \widehat{C}(\sigma)$, from* (5.30) *is infinitely often differentiable.*

PROOF. This follows along the lines of the proof of Lemma 3.4.1. □

Theorem 5.4.8 (Second-order approximation to implied volatility)**.** *Assume that for the function $\widehat{C}$ from* (5.30)*, we have $\widehat{C}'(\sigma) \neq 0$ for all $\sigma > 0$ and that $\{c^\lambda : \lambda \in [0,1]\} \subseteq \widehat{C}((0,\infty))$ for the option c^λ from* (5.22) *relative to S^λ. For $\lambda \in [0,1]$, define the implied volatility of the option price c^λ by*

$$\sigma^\lambda_{\text{impl}} := \widehat{C}^{-1}(c^\lambda). \tag{5.31}$$

Then, $\lambda \mapsto \sigma^\lambda_{\text{impl}}$ is in $C^\infty([0,1],\mathbb{R}_+)$, and the second-order approximation to $\sigma_{\text{impl}} := \sigma^1_{\text{impl}}$ corresponding to Definition 5.2.28 is given by

$$\mathfrak{A}\left(\sigma_{\text{impl}}\right) = \mathfrak{A}_0\left(\sigma_{\text{impl}}\right) + \mathfrak{A}_1\left(\sigma_{\text{impl}}\right) + \frac{1}{2}\mathfrak{A}_2\left(\sigma_{\text{impl}}\right)$$

with

$$\begin{aligned}
\mathfrak{A}_0\left(\sigma_{\text{impl}}\right) &= \overline{\sigma}, \\
\mathfrak{A}_1\left(\sigma_{\text{impl}}\right) &= \frac{1}{\widehat{C}'(\overline{\sigma})}\mathfrak{A}_1(c), \\
\mathfrak{A}_2\left(\sigma_{\text{impl}}\right) &= -\frac{\widehat{C}''(\overline{\sigma})}{\left(\widehat{C}'(\overline{\sigma})\right)^3}\left(\mathfrak{A}_1(c)\right)^2 + \frac{1}{\widehat{C}'(\overline{\sigma})}\mathfrak{A}_2(c)
\end{aligned} \tag{5.32}$$

for function $\widehat{C}$ *from* (5.30)*,* $\mathfrak{A}_1(c)$ *and* $\mathfrak{A}_2(c)$ *from Theorem 5.4.3, and* $\overline{\sigma}$ *from* (5.16)*. Moreover, we have the relations*

$$\begin{aligned}
\widehat{C}'(\overline{\sigma}) &= \overline{\sigma} T D_2(S_0), \\
\widehat{C}''(\overline{\sigma}) &= T D_2(S_0) + \overline{\sigma}^2 T^2 (2D_2(S_0) + 4D_3(S_0) + D_4(S_0))
\end{aligned} \tag{5.33}$$

for the cash greeks $D_k(S_0)$*,* $k = 1, \ldots, 4$*, as in Definition 5.4.2.*

PROOF. Cf. Section 5.7.3.9. □

Example 5.4.9. We illustrate our approximation to implied volatility from Theorem 5.4.8 for a European call option with discounted strike $K > 0$ in a geometric Lévy model, i.e., $V_t = t$ and $U_t = 0$ for $t \in \mathbb{R}_+$. Note that for the corresponding payoff function $f(s) = (s-K)^+$, $s \in \mathbb{R}_+$, there is an integral representation by Example 3.2.3, and we we have $\widehat{C}'(\sigma) > 0$ for all $\sigma > 0$ by Representation (5.33) and the closed-form expressions for greeks of the European call option from Appendix D. Moreover, it is well known that in the case of a call option

$$\lim_{\sigma\to 0}\widehat{C}(\sigma) = (S_0 - K)^+ \quad \text{and} \quad \lim_{\sigma\to\infty}\widehat{C}(\sigma) = S_0.$$

By Jensen's inequality, we have for all $\lambda \in [0,1]$

$$(S_0 - K)^+ = \left(\mathrm{E}\left(S_T^\lambda\right) - K\right)^+ \leq \mathrm{E}\left(\left(S_T^\lambda - K\right)^+\right) = c^\lambda \leq S_0.$$

It is easy to see that the inequalities are equalities only in degenerate cases, and hence it is reasonable to assume that $\{c^\lambda : \lambda \in [0,1]\} \subseteq \widehat{C}((0,\infty))$. For the relevant quantities of the geometric Lévy model, we consider

$$S_0 = 100, \quad \operatorname{Var}(L_1) = 0.4^2, \quad \operatorname{Skew}(L_1) \in \left\{0, \frac{-4}{\sqrt{250}}\right\}, \quad \operatorname{ExKurt}(L_1) = \frac{60}{250}.$$

These moments are within the plausible range for a risk-neutral probability measure obtained from calibration to observed prices, cf. [CGMY02]. Figure 5.1 and Figure 5.2 show the approximate implied volatility as in Theorem 5.4.8 in dependence on different strikes and maturities for $\operatorname{Skew}(L_1) = 0$ and $\operatorname{Skew}(L_1) = \frac{-4}{\sqrt{250}}$. In the case of zero skewness, we observe the smile, in the case of pronounced negative skewness the skew of implied volatilities. Both phenomena are less prevalent for longer maturities. Depending on the underlying, one of these shapes for implied volatility surfaces is typically observed on real markets. Hence, this experiment shows that our approximate option pricing formula is able to capture these empirical phenomena.

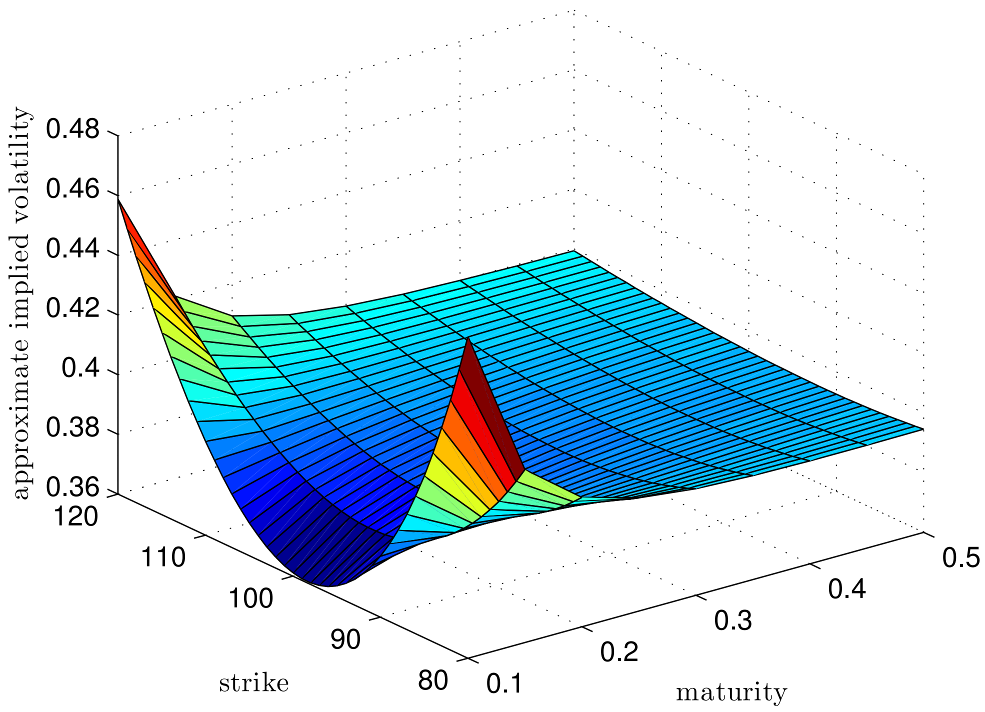

Figure 5.1.: Second-order approximation to implied volatility for call options in a geometric Lévy model with $S_0 = 100$, $\mathrm{Var}(L_1) = 0.4^2$, $\mathrm{Skew}(L_1) = 0$, $\mathrm{ExKurt}(L_1) = \frac{60}{250}$.

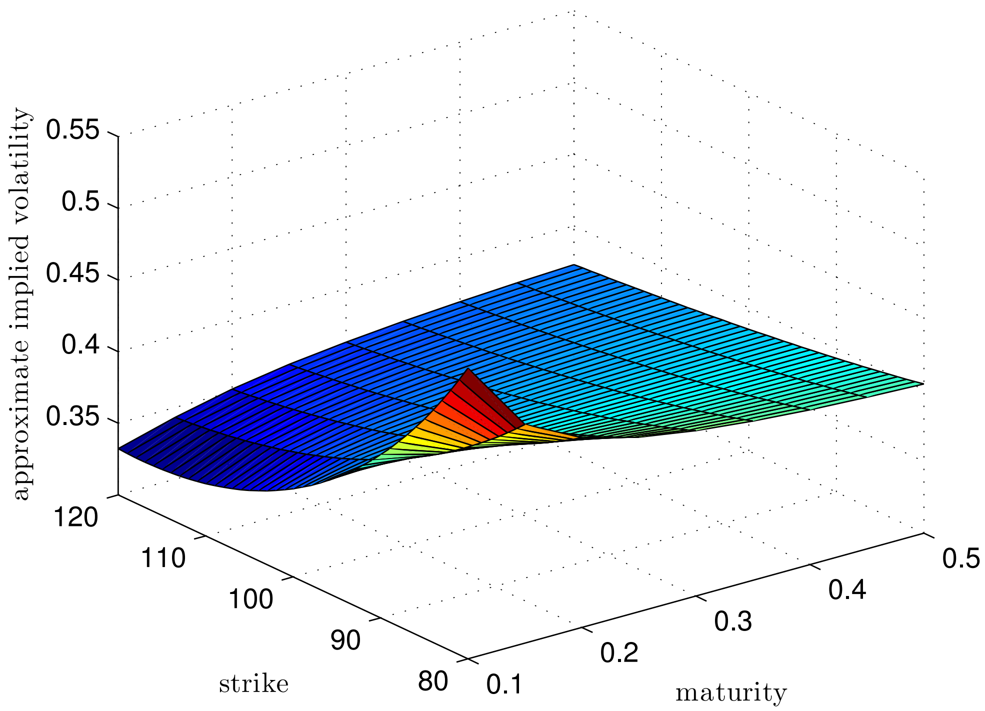

Figure 5.2.: Second-order approximation to implied volatility for call options in a geometric Lévy model with $S_0 = 100$, $\mathrm{Var}(L_1) = 0.4^2$, $\mathrm{Skew}(L_1) = \frac{-4}{\sqrt{250}}$, $\mathrm{ExKurt}(L_1) = \frac{60}{250}$.

5.5. Examples and numerical tests

In this section, we illustrate our approximation in four parametric models from the literature, namely the Heston model [Hes93] and the extended Stein & Stein model [SS91, SZ99] as examples for bivariate diffusion models (cf. Section 5.3.2), the Merton model with normal jumps [Mer76] as example for a geometric Lévy model (cf. Section 5.3.1), and the NIG-CIR model as an example for a model of the class suggested by [CGMY03] (cf. Section 5.3.4).

For every parametric model, we provide an exact specification and demonstrate that it can be interpreted as a stochastic volatility model in the sense of Definition 5.2.1. We verify that the regularity conditions necessary four our analysis (cf. Table 5.1) are either always satisfied, or we provide easy-to-check conditions on the model parameters. Moreover, we derive representations of the moments used in our approximation from Theorem 5.4.3 in terms of explicit functions of the model parameters. Finally, we assess the quality of our approximation for different parameter sets from the literature, considering European call options for a broad set of maturities and strikes. We compare our approximate prices with approximations from the literature.

5.5.1. General setup of the numerical tests

5.5.1.1. Option

We assess our approximation for European call options with maturities

$$T = \frac{1}{12},\ \frac{1}{4},\ \frac{1}{2},\ 1,\ 2,\ 4,\ 8,\ 16 \ \text{ years}$$

in the Heston, the extended Stein & Stein, and the NIG-CIR model resp.

$$T = \frac{1}{12},\ \frac{5}{12},\ 1.5,\ 3 \ \text{ years}$$

in the Merton model since in this case the parameters are obtained by calibration to options with these maturities.

For every fixed maturity T, we consider 9 different strikes, i.e., 9 different call options. Due to the broad range of possible maturities, we choose these strikes in dependence on the respective maturity in order to obtain reasonable results. More specifically, for the maturity T we consider the strikes

$$K_i \approx e^{\left(r-\frac{1}{2}\sigma^2\right)T+q\widetilde{\sigma}\sqrt{T}\left(\frac{-5+i}{9}\right)}, \quad i=1,\dots,9,$$

where $q = 2.5758$ is the 99.5%-quantile of the standard normal distribution. Hence, the strikes are chosen such that in a Black-Scholes model with interest rate $r > 0$ and volatility parameter $\widetilde{\sigma} > 0$ the risk-neutral stock price process undershoots or overshoots the extreme strikes at maturity only with probability 1%. By $\approx$ we denote suitable rounding to values divisible by 5. A reasonable value for $\widetilde{\sigma}$ is chosen in dependence on every parameter set.

Recall from Example 3.2.3 that the payoff function $s \mapsto (s-K)^+$ of a European call option with strike $K>0$ allows for the integral representation

$$(s-K)^+ = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)}\,\mathrm{d}z, \quad s \in \mathbb{R}_+, \tag{5.34}$$

where R can be chosen arbitrarily in $(1,\infty)$. In the case of a non-smooth payoff function, such representation is essential to our analysis, cf. Assumption 5.2.21. Moreover, cash greeks of European call options in the Black-Scholes model – as they appear in our approximation from Theorem 5.4.3 – are available in closed form and can hence be easily computed. We present these formulas in Appendix D.

5.5.1.2. Provided figures and generic benchmarks

For every choice of parameters in every model class, we provide one table for every maturity under consideration. For every corresponding strike, we provide the exact option price (Exact) and our approximation (D), as well as approximations from the literature that are specified below in the concrete cases. In round brackets, we report the error of the respective approximation relative to S_0 in percent, i.e.,

$$\frac{\text{Approximate price} - \text{Exact price}}{S_0} \cdot 100.$$

Since we always choose $S_0 = 100$, this amounts to the absolute error. The values in round brackets below the abbreviations of the different approximations in the head of a table indicate the mean absolute error over all strikes (MAEOS), i.e.,

$$\frac{\frac{1}{9}\sum_{i=1}^{9} |\text{Approximate price}(K_i) - \text{Exact price}(K_i)|}{S_0} \cdot 100.$$

This aggregate criterion allows to quickly compare the quality of different approximations for a common maturity.

For all parametric model classes, we consider two generic approximations as benchmarks:

1. The Black-Scholes price (BS) of the respective option relative to the volatility

$$\overline{\sigma} = \sqrt{\frac{1}{T}\left(\mathrm{E}\left(V_T\right)\mathrm{Var}\left(L_1\right) + \mathrm{Var}\left(M_T\right)\right)}$$

   from (5.16). This corresponds to the zero-order term in our approximation.

2. The approximation from [JR82] (JR), which can be evaluated in quite general situations. The approach is discussed in detail in Section 5.6.1.

5.5.1.3. Computation of exact prices

In order to compute the exact option prices in the respective model, we evaluate the integral representation of the price from Theorem 3.1.1 numerically. This requires that the extended characteristic function of $\log(S_T)$ can be efficiently evaluated. We state references for a closed-form representation of the respective characteristic function in the sections on the different parametric models. For the numerical quadrature of the price integral, we always use $R = 2.1$ and the quadrature routine `quadgk` of the software package MATLAB®.

5.5.2. Heston model

5.5.2.1. Model specification

In the Heston model [Hes93], the logarithmic stock price $X = \log(S)$ is given by

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2} y_t \,\mathrm{d}t + \sqrt{y_t}\left(\rho\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2\right), \quad X_0 = \log(S_0), \\ \mathrm{d}y_t &= \kappa(\eta - y_t)\,\mathrm{d}t + \theta\sqrt{y_t}\,\mathrm{d}W_t^1, \quad y_0 > 0, \end{aligned} \tag{5.35}$$

for independent standard Brownian motions W^1, W^2, $\rho \in [-1,1]$, κ, η, $\theta > 0$.

The following lemma provides the regularity condition required to apply Proposition 5.3.2, which allows us to interpret the model from (5.35) as stochastic volatility model (S_0, L, V, U) in the sense of Definition 5.2.1.

Lemma 5.5.1. *In the Heston model* (5.35), *we have* $\int_0^\infty y_t \,\mathrm{d}t = \infty$.

PROOF. Cf. Section 5.7.4. □

Remark 5.5.2. In the proof of Lemma 5.5.1, we use the lucky fact that the Laplace transform of integrated instantaneous variance is available in explicit form. If this is not the case in other situations, one can exploit that the processes used to model instantaneous volatility or variance typically possess a stationary distribution and are ergodic. Let us exemplify this strategy in the Heston model, where the stationary distribution of y is a Gamma distribution with mean η and variance $\frac{\eta\theta^2}{2\kappa}$ (cf. [CI85, Section 3]). Denoting its cumulative distribution function by $\Gamma_{\kappa,\eta,\theta}$, one has

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(y_s)\,\mathrm{d}s = \int_{\mathbb{R}} g(x)\,\mathrm{d}\Gamma_{\kappa,\eta,\theta}(x) \quad \text{almost surely}$$

for every bounded and measurable function $g : \mathbb{R}_+ \to \mathbb{R}$. Setting, e.g., $g(x) = (|x| \wedge 1)$, we see that $\frac{1}{t}\int_0^t g(y_s)\,\mathrm{d}s$ almost surely has a non-negative limit, and hence $\lim_{t\to\infty}\int_0^t g(y_s)\,\mathrm{d}s = \infty$ almost surely. Since $y \geq g(y)$, we also have $\lim_{t\to\infty}\int_0^t y_s\,\mathrm{d}s = \infty$ almost surely.

Hence, for two independent standard Brownian motions W^1, W^2, parameters $\rho \in [-1,1]$, κ, η, $\theta > 0$, and the process y as in (5.35), we can consider the Heston model to be generated by a stochastic volatility model in the sense of Definition 5.2.1 for some initial stock price $S_0 > 0$ and

$$\begin{aligned} L_t &= \sqrt{1-\rho^2}W_t^2 - \frac{1}{2}(1-\rho^2)t, \\ V_t &= \int_0^t y_s \,\mathrm{d}s, \\ M_t &= \rho \int_0^t \sqrt{y_s}\,\mathrm{d}W_s^1, \\ U_t &= M_t - \frac{1}{2}\rho^2 \int_0^t y_s \,\mathrm{d}s \end{aligned} \tag{5.36}$$

for $t \in \mathbb{R}_+$.

It is part of Definition 5.2.1 that the process V is almost surely strictly increasing. By the next lemma, this is indeed the case for the Heston model.

Lemma 5.5.3. *In the Heston model* (5.36)*, the mapping $t \mapsto \int_0^t y_s \,\mathrm{d}s$ is almost surely strictly increasing.*

PROOF. Cf. Section 5.7.4. □

5.5.2.2. Check of regularity conditions

We now verify that the assumptions necessary for our analysis (cf. Table 5.1) hold in general in the Heston model, or we provide sufficient criteria on the model parameters.

Since L_1 is normally distributed, Assumptions 5.2.4 and 5.2.21(2) obviously hold. Assumption 5.2.9 clearly holds as well. Finally, Assumptions 5.2.16 and 5.2.26 are no restrictions since M is continuous.

Lemma 5.5.4 (Assumption 5.2.8)**.** *In the Heston model* (5.36)*, M is a martingale.*

PROOF. For all $t \in \mathbb{R}_+$, we have $\mathrm{E}\left(\rho^2 \int_0^t y_s\, ds\right) = \rho^2 \mathrm{E}\left(V_t\right) < \infty$ by Lemma 5.5.9 below, which implies that the local martingale M is a martingale. □

Lemma 5.5.5 (Assumption 5.2.11)**.** *In the Heston model* (5.36)*, the process e^{U^λ} defined in* (5.12) *is a martingale for all $\lambda \in [0,1]$.*

PROOF. Cf. Section 5.7.4. □

Lemma 5.5.6 (Assumption 5.2.21(1a))**.** *In the Heston model* (5.36)*, for all $t \in \mathbb{R}_+$, there exist b_1, $b_2 > 0$ such that*

$$\mathrm{E}\left(e^{uV_t}\right) \le e^{b_1 - b_2\sqrt{|u|}} \quad \textit{for all } u \in \mathbb{R}_-.$$

PROOF. Cf. Section 5.7.4. □

The next lemma provides a criterion to check Assumption 5.2.21(3), which depends also on $\kappa^\lambda(R)$. Note that since L is a Brownian motion with drift in the Heston model, we have by construction

$$\kappa^\lambda(z) = \kappa(z) = \frac{1}{2}(1-\rho^2)z(z-1) \quad \text{for all } \lambda \in [0,1],\ z \in \mathbb{C}. \tag{5.37}$$

Lemma 5.5.7 (Criterion for Assumption 5.2.21(3))**.** *In the Heston model* (5.36)*, for all $t \in \mathbb{R}_+$ and for all $a \leq \frac{\kappa^2}{2\theta^2}$, it holds* $\mathrm{E}\left(e^{aV_t}\right) < \infty$.

PROOF. Cf. [AP07, Corollary 3.3]. $\square$

Lemma 5.5.8 (Criterion for Assumption 5.2.21(4))**.** *In the Heston model* (5.36)*, for all $t \in \mathbb{R}_+$ and all $a \in \mathbb{R}$ such that $a\rho < \frac{\kappa}{2\theta}$ we have*

$$\mathrm{E}\left(e^{aM_t}\right) < \infty.$$

Moreover, for all $t \in \mathbb{R}_+$ and all $b \in \mathbb{R}$ such that $2b\rho < \frac{\kappa}{2\theta}$ and $-b\rho^2 \leq \frac{\kappa^2}{2\theta^2}$, we have

$$\mathrm{E}\left(e^{bU_t}\right) < \infty.$$

PROOF. Cf. Section 5.7.4. $\square$

Lemma 5.5.9 (Assumptions 5.2.15 and 5.2.25)**.** *In the Heston model* (5.36)*, for all $t \in \mathbb{R}_+$, the random variables M_t, $\langle M,M\rangle_t$, and V_t have moments of any order.*

PROOF. By Lemma 5.5.8, there is $\varepsilon > 0$ such that for all $t \in \mathbb{R}_+$, it holds $\mathrm{E}\left(e^{\pm\varepsilon M_t}\right) < \infty$. This implies the existence of all moments of M_t. By Lemma 5.5.7, we can apply the same argument to $\langle M,M\rangle_t = \rho^2 V_t$ and V_t, $t \in \mathbb{R}_+$. $\square$

5.5.2.3. Moments required in the approximation

In the case of the Heston model (5.36), the moments used in our approximate option pricing formula from Theorem 5.4.3 are given by

$$\begin{aligned}
\mathrm{Var}(L_1) &= 1-\rho^2,\\
\mathrm{Skew}(L_1) &= 0,\\
\mathrm{ExKurt}(L_1) &= 0,\\
\mathrm{E}(V_T) &= \eta T + \frac{(y_0-\eta)(1-e^{-\kappa T})}{\kappa},\\
\mathrm{Var}(V_T) &= \frac{\theta^2}{2\kappa^3}\Big(2y_0 - 5\eta - 2y_0 e^{-2\kappa T} - 4y_0 e^{-\kappa T}\kappa T\\
&\qquad + \eta e^{-2\kappa T} + 4\eta e^{-\kappa T}(\kappa T+1) + 2\eta T\kappa\Big),\\
\mathrm{Var}(M_T) &= \rho^2 \mathrm{E}(V_T),\\
\mathrm{Cov}(V_T, M_T) &= \frac{\rho}{\theta}\left(\int_0^T \mathrm{Cov}(y_T, y_t)\,\mathrm{d}t + \kappa \mathrm{Var}(V_T)\right),
\end{aligned}$$

S_0	r	y_0	κ	η	θ	ρ
100	0	0.04	3.00	0.06	0.30	0

Table 5.2.: Heston model parameters (zero correlation)

S_0	r	y_0	κ	η	θ	ρ
100	0	0.04	3.00	0.06	0.30	−0.2

Table 5.3.: Heston model parameters (medium correlation)

where

$$\int_0^T \mathrm{Cov}\left(y_T, y_t\right) \mathrm{d}t = \frac{\theta^2}{2\kappa^2}\left(2e^{-\kappa T}(y_0\kappa T - y_0 - \eta\kappa T) + e^{-2\kappa T}(2y_0 - \eta) + \eta\right).$$

The derivation is delegated to Appendix B.1.

5.5.2.4. Numerical comparison

We asses the quality of our approximation in the Heston model for three different parameter sets given in Tables 5.2, 5.3, and 5.4. Besides our generic benchmarks BS and JR (cf. Section 5.5.1.2), we compare our approximation to those of [Alò12], indicated by A, and [BGM10b] resp. [Lew00], indicated by BGM/L. We discuss these approaches in Sections 5.6.7 and 5.6.5.

All three parameter sets are taken from [BGM10b]; they differ only by the correlation parameter ρ, which we choose to be 0 (Table 5.2), -0.2 (Table 5.3), and -0.5 (Table 5.4).

In order to determine reasonable strikes for every maturity, we use the Black-Scholes volatility $\widetilde{\sigma} = \sqrt{\eta}$, cf. Section 5.5.1.1. The value $\sqrt{\eta}$ corresponds to the long-term mean volatility in the Heston model, i.e., $\lim_{t\to\infty}\sqrt{\frac{1}{t}\int_0^t y_s\,\mathrm{d}s} = \sqrt{\eta}$.

We use the representation of the characteristic function of $X_T = \log(S_T)$ from [LK10] to compute the exact option price by the Laplace transform method (cf. Section 5.5.1.3).

As we see from Section 5.5.2.2, the only regularity conditions in the Heston model whose validity depends on the choice of the parameters are Assumptions 5.2.21(3) and 5.2.21(4). In the integral

S_0	r	y_0	κ	η	θ	ρ
100	0	0.04	3.00	0.06	0.30	−0.5

Table 5.4.: Heston model parameters (high correlation)

representation of the call payoff function from (5.34), we may choose $R \in (1,\infty)$ arbitrarily, and hence R can be chosen such that $\kappa^{\lambda}(R) = \frac{1}{2}(1-\rho^2)R(R-1)$ is arbitrarily close to 0. By Lemma 5.5.7, Assumption 5.2.21(3) may then always be satisfied in the case of a European call option. Since we are in the situation that $R > 1$ and $\rho \leq 0$, Assumption 5.2.21(4) is satisfied by Lemma 5.5.8.

Tables 5.5 and 5.6 show the exact price, the different approximations, and their errors for the parameters from Table 5.2 ($\rho = 0$), Tables 5.7 and 5.8 for the parameters from Table 5.3 ($\rho = -0.2$), and Tables 5.9 and 5.10 for the parameters from Table 5.4 ($\rho = -0.5$).

In the case $\rho = 0$, our approximation coincides with those of A and BGM/L, which is why we obtain identical figures. The MAEOS is 0 for the shortest maturity $T = \frac{1}{12}$, slightly increases to a level of 0.002 for medium maturities, and decreases to 0 for the longer maturities $T = 8$ and $T = 16$. Probably, this can be explained by the fact that stochastic volatility does not vary significantly up to the short maturity $T = \frac{1}{12}$, but it does for medium maturities. For longer maturities, the stochastic volatility process reaches probably already its stationary regime. We observe that our approximation significantly improves the mere Black-Scholes price BS and, in particular for medium and long maturities, the approximation JR.

For $\rho \neq 0$, our approximation and those of A and BGM/L all differ. In the case $\rho = -0.2$, the MAEOS of our approximation is 0 for $T = \frac{1}{12}$, increases to 0.005 up to $T = 2$, and stays at this level for the remaining maturities. In terms of the MAEOS, our approximation significantly outperforms BS and JR, and it is roughly at the same level as the MAEOS of A and BGM/L up to the maturity $T = 4$. For $T = 16$, the MAEOS of BGM/L amounts to 0.001, while ours is 0.005.

For $\rho = 0.5$, the MAEOS of our approximation amounts to 0.007 for $T = \frac{1}{12}$, and it increases to a level of 0.100 for $T = 16$. In terms of the MAEOS, our approximation outperforms the mere Black-Scholes price BS for all maturities, and it is superior to JR for maturities from $T = 1$ onwards. The approximations A and BGM/L, which are tailor-made for the Heston model, clearly outperform our approximation for all maturities. However, given that our approximation is valid in a much more general framework and that it can be interpreted as an expansion around $\rho = 0$, the results in the case $\rho = -0.5$ are surprisingly reasonable.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.004)	A (0.000)	BGM/L (0.000)	D (0.000)	JR (0.002)
80	20.000 (0.000)	20.000 (-0.000)	20.000 (-0.000)	20.000 (-0.000)	20.000 (-0.000)	20.000 (0.000)
85	15.007 (0.000)	15.005 (-0.002)	15.007 (0.000)	15.007 (0.000)	15.007 (0.000)	15.009 (0.001)
90	10.091 (0.000)	10.086 (-0.006)	10.092 (0.000)	10.092 (0.000)	10.092 (0.000)	10.092 (0.001)
95	5.618 (0.000)	5.621 (0.002)	5.618 (-0.000)	5.618 (-0.000)	5.618 (-0.000)	5.613 (-0.005)
100	2.354 (0.000)	2.368 (0.014)	2.354 (-0.000)	2.354 (-0.000)	2.354 (-0.000)	2.354 (-0.000)
105	0.700 (0.000)	0.704 (0.003)	0.700 (-0.000)	0.700 (-0.000)	0.700 (-0.000)	0.705 (0.004)
110	0.149 (0.000)	0.143 (-0.006)	0.149 (0.000)	0.149 (0.000)	0.149 (0.000)	0.150 (0.001)
115	0.024 (0.000)	0.020 (-0.004)	0.024 (0.000)	0.024 (0.000)	0.024 (0.000)	0.023 (-0.001)
120	0.003 (0.000)	0.002 (-0.001)	0.003 (-0.000)	0.003 (-0.000)	0.003 (-0.000)	0.003 (-0.000)

(b) $T = \frac{1}{4}$

K	Exact (0.000)	BS (0.018)	A (0.001)	BGM/L (0.001)	D (0.001)	JR (0.012)
70	30.003 (0.000)	30.001 (-0.002)	30.003 (-0.000)	30.003 (-0.000)	30.003 (-0.000)	30.008 (0.005)
80	20.081 (0.000)	20.065 (-0.016)	20.082 (0.001)	20.082 (0.001)	20.082 (0.001)	20.094 (0.014)
85	15.296 (0.000)	15.278 (-0.018)	15.297 (0.001)	15.297 (0.001)	15.297 (0.001)	15.288 (-0.008)
90	10.873 (0.000)	10.874 (0.000)	10.873 (-0.001)	10.873 (-0.001)	10.873 (-0.001)	10.838 (-0.035)
100	4.225 (0.000)	4.273 (0.048)	4.224 (-0.001)	4.224 (-0.001)	4.224 (-0.001)	4.220 (-0.005)
105	2.291 (0.000)	2.325 (0.035)	2.290 (-0.001)	2.290 (-0.001)	2.290 (-0.001)	2.312 (0.021)
115	0.533 (0.000)	0.518 (-0.015)	0.534 (0.000)	0.534 (0.000)	0.534 (0.000)	0.549 (0.015)
125	0.101 (0.000)	0.081 (-0.019)	0.102 (0.001)	0.102 (0.001)	0.102 (0.001)	0.098 (-0.003)
135	0.017 (0.000)	0.009 (-0.008)	0.017 (0.000)	0.017 (0.000)	0.017 (0.000)	0.013 (-0.004)

(c) $T = \frac{1}{2}$

K	Exact (0.000)	BS (0.031)	A (0.002)	BGM/L (0.002)	D (0.002)	JR (0.045)
60	40.006 (0.000)	40.002 (-0.004)	40.005 (-0.001)	40.005 (-0.001)	40.005 (-0.001)	40.029 (0.024)
70	30.074 (0.000)	30.053 (-0.021)	30.076 (0.002)	30.076 (0.002)	30.076 (0.002)	30.129 (0.055)
80	20.523 (0.000)	20.497 (-0.026)	20.525 (0.002)	20.525 (0.002)	20.525 (0.002)	20.477 (-0.046)
90	12.214 (0.000)	12.247 (0.033)	12.212 (-0.002)	12.212 (-0.002)	12.212 (-0.002)	12.093 (-0.121)
100	6.197 (0.000)	6.279 (0.081)	6.195 (-0.003)	6.195 (-0.003)	6.195 (-0.003)	6.179 (-0.018)
110	2.716 (0.000)	2.760 (0.044)	2.713 (-0.003)	2.713 (-0.003)	2.713 (-0.003)	2.783 (0.067)
120	1.069 (0.000)	1.054 (-0.015)	1.070 (0.000)	1.070 (0.000)	1.070 (0.000)	1.123 (0.053)
130	0.395 (0.000)	0.356 (-0.039)	0.398 (0.003)	0.398 (0.003)	0.398 (0.003)	0.409 (0.014)
150	0.051 (0.000)	0.031 (-0.020)	0.052 (0.001)	0.052 (0.001)	0.052 (0.001)	0.041 (-0.009)

(d) $T = 1$

K	Exact (0.000)	BS (0.040)	A (0.002)	BGM/L (0.002)	D (0.002)	JR (0.122)
50	50.014 (0.000)	50.006 (-0.007)	50.013 (-0.000)	50.013 (-0.000)	50.013 (-0.000)	50.165 (0.151)
60	40.112 (0.000)	40.086 (-0.026)	40.114 (0.002)	40.114 (0.002)	40.114 (0.002)	40.289 (0.177)
70	30.551 (0.000)	30.516 (-0.035)	30.554 (0.003)	30.554 (0.003)	30.554 (0.003)	30.450 (-0.101)
80	21.838 (0.000)	21.843 (0.005)	21.838 (-0.001)	21.838 (-0.001)	21.838 (-0.001)	21.497 (-0.341)
100	9.115 (0.000)	9.221 (0.106)	9.111 (-0.004)	9.111 (-0.004)	9.111 (-0.004)	9.070 (-0.045)
110	5.402 (0.000)	5.486 (0.084)	5.397 (-0.004)	5.397 (-0.004)	5.397 (-0.004)	5.522 (0.121)
130	1.708 (0.000)	1.691 (-0.017)	1.709 (0.001)	1.709 (0.001)	1.709 (0.001)	1.830 (0.122)
150	0.510 (0.000)	0.456 (-0.054)	0.515 (0.005)	0.515 (0.005)	0.515 (0.005)	0.531 (0.021)
180	0.086 (0.000)	0.055 (-0.030)	0.087 (0.001)	0.087 (0.001)	0.087 (0.001)	0.069 (-0.016)

Table 5.5.: Exact and approximated option prices with errors for different strikes K and maturities T in the Heston model for parameters as in Table 5.2 (zero correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò12], **BGM/L** to the approximation by [BGM10b] resp. [Lew01], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = 2$

K	Exact (0.000)	BS (0.046)	A (0.002)	BGM/L (0.002)	D (0.002)	JR (0.372)
40	60.031 (0.000)	60.021 (-0.011)	60.031 (0.000)	60.031 (0.000)	60.031 (0.000)	60.894 (0.863)
50	50.198 (0.000)	50.170 (-0.028)	50.200 (0.002)	50.200 (0.002)	50.200 (0.002)	50.562 (0.363)
60	40.759 (0.000)	40.727 (-0.031)	40.761 (0.002)	40.761 (0.002)	40.761 (0.002)	40.176 (-0.583)
80	24.501 (0.000)	24.544 (0.044)	24.499 (-0.002)	24.499 (-0.002)	24.499 (-0.002)	23.595 (-0.906)
100	13.261 (0.000)	13.368 (0.107)	13.257 (-0.004)	13.257 (-0.004)	13.257 (-0.004)	13.211 (-0.050)
120	6.711 (0.000)	6.783 (0.072)	6.708 (-0.003)	6.708 (-0.003)	6.708 (-0.003)	7.035 (0.324)
150	2.296 (0.000)	2.271 (-0.025)	2.298 (0.002)	2.298 (0.002)	2.298 (0.002)	2.499 (0.202)
180	0.790 (0.000)	0.730 (-0.059)	0.794 (0.004)	0.794 (0.004)	0.794 (0.004)	0.829 (0.040)
230	0.147 (0.000)	0.110 (-0.038)	0.149 (0.001)	0.149 (0.001)	0.149 (0.001)	0.126 (-0.021)

(b) $T = 4$

K	Exact (0.000)	BS (0.035)	A (0.001)	BGM/L (0.001)	D (0.001)	JR (2.357)
20	80.004 (0.000)	80.002 (-0.002)	80.004 (-0.000)	80.004 (-0.000)	80.004 (-0.000)	85.924 (5.920)
30	70.064 (0.000)	70.053 (-0.011)	70.064 (0.000)	70.064 (0.000)	70.064 (0.000)	74.188 (4.125)
50	51.156 (0.000)	51.135 (-0.021)	51.157 (0.001)	51.157 (0.001)	51.157 (0.001)	46.399 (-4.757)
60	42.741 (0.000)	42.739 (-0.002)	42.741 (0.000)	42.741 (0.000)	42.741 (0.000)	38.071 (-4.669)
100	18.994 (0.000)	19.085 (0.091)	18.993 (-0.002)	18.993 (-0.002)	18.993 (-0.002)	19.259 (0.265)
120	12.358 (0.000)	12.438 (0.080)	12.356 (-0.002)	12.356 (-0.002)	12.356 (-0.002)	13.184 (0.826)
160	5.253 (0.000)	5.261 (0.008)	5.253 (-0.000)	5.253 (-0.000)	5.253 (-0.000)	5.813 (0.560)
230	1.290 (0.000)	1.235 (-0.055)	1.292 (0.002)	1.292 (0.002)	1.292 (0.002)	1.364 (0.074)
310	0.311 (0.000)	0.268 (-0.043)	0.312 (0.001)	0.312 (0.001)	0.312 (0.001)	0.291 (-0.020)

(c) $T = 8$

K	Exact (0.000)	BS (0.030)	A (0.000)	BGM/L (0.000)	D (0.000)	JR (35.819)
10	90.003 (0.000)	90.002 (-0.001)	90.003 (-0.000)	90.003 (-0.000)	90.003 (-0.000)	261.17 (171.2)
20	80.104 (0.000)	80.096 (-0.008)	80.104 (0.000)	80.104 (0.000)	80.104 (0.000)	56.573 (-23.53)
30	70.597 (0.000)	70.583 (-0.014)	70.598 (0.000)	70.598 (0.000)	70.598 (0.000)	-9.073 (-79.67)
50	53.846 (0.000)	53.848 (0.003)	53.846 (-0.000)	53.846 (-0.000)	53.846 (-0.000)	18.298 (-35.55)
100	26.844 (0.000)	26.915 (0.071)	26.844 (-0.001)	26.844 (-0.001)	26.844 (-0.001)	32.067 (5.222)
120	20.526 (0.000)	20.597 (0.070)	20.526 (-0.001)	20.526 (-0.001)	20.526 (-0.001)	25.666 (5.139)
190	8.627 (0.000)	8.641 (0.014)	8.626 (-0.000)	8.626 (-0.000)	8.626 (-0.000)	10.448 (1.821)
300	2.715 (0.000)	2.671 (-0.043)	2.716 (0.001)	2.716 (0.001)	2.716 (0.001)	2.977 (0.263)
470	0.646 (0.000)	0.603 (-0.043)	0.647 (0.001)	0.647 (0.001)	0.647 (0.001)	0.636 (-0.009)

(d) $T = 16$

K	Exact (0.000)	BS (0.023)	A (0.000)	BGM/L (0.000)	D (0.000)	JR (2976)
5	95.007 (0.000)	95.006 (-0.001)	95.007 (-0.000)	95.007 (-0.000)	95.007 (-0.000)	-228.47 (-323.5)
10	90.090 (0.000)	90.086 (-0.004)	90.090 (0.000)	90.090 (0.000)	90.090 (0.000)	-17486 (-17576)
20	80.835 (0.000)	80.828 (-0.007)	80.836 (0.000)	80.836 (0.000)	80.836 (0.000)	-6708 (-6789)
30	72.588 (0.000)	72.585 (-0.003)	72.588 (0.000)	72.588 (0.000)	72.588 (0.000)	-1782 (-1855)
100	37.405 (0.000)	37.459 (0.054)	37.404 (-0.000)	37.404 (-0.000)	37.404 (-0.000)	255.74 (218.3)
220	15.988 (0.000)	16.016 (0.027)	15.988 (-0.000)	15.988 (-0.000)	15.988 (-0.000)	40.994 (25.01)
410	6.051 (0.000)	6.025 (-0.026)	6.051 (0.000)	6.051 (0.000)	6.051 (0.000)	8.320 (2.269)
770	1.679 (0.000)	1.636 (-0.042)	1.679 (0.001)	1.679 (0.001)	1.679 (0.001)	1.770 (0.092)
850	1.335 (0.000)	1.294 (-0.041)	1.335 (0.000)	1.335 (0.000)	1.335 (0.000)	1.378 (0.043)

Table 5.6.: Exact and approximated option prices with errors for different strikes K and maturities T in the Heston model for parameters as in Table 5.2 (zero correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò12], **BGM/L** to the approximation by [BGM10b] resp. [Lew01], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.013)	A (0.000)	BGM/L (0.000)	D (0.000)	JR (0.001)
80	20.001 (0.000)	20.000 (-0.000)	20.000 (-0.000)	20.000 (-0.000)	20.000 (-0.000)	20.001 (-0.000)
85	15.010 (0.000)	15.005 (-0.005)	15.010 (-0.000)	15.010 (-0.000)	15.010 (-0.001)	15.011 (0.001)
90	10.106 (0.000)	10.086 (-0.021)	10.107 (0.001)	10.107 (0.001)	10.106 (-0.000)	10.108 (0.001)
95	5.643 (0.000)	5.621 (-0.022)	5.643 (0.001)	5.643 (0.000)	5.643 (-0.000)	5.640 (-0.003)
100	2.353 (0.000)	2.368 (0.015)	2.353 (-0.000)	2.353 (-0.000)	2.354 (0.001)	2.352 (-0.001)
105	0.672 (0.000)	0.704 (0.032)	0.672 (0.000)	0.672 (-0.000)	0.673 (0.001)	0.675 (0.003)
110	0.128 (0.000)	0.143 (0.014)	0.128 (-0.000)	0.128 (-0.001)	0.128 (-0.000)	0.129 (0.000)
115	0.017 (0.000)	0.020 (0.003)	0.017 (-0.000)	0.017 (0.000)	0.017 (-0.000)	0.017 (-0.001)
120	0.002 (0.000)	0.002 (0.000)	0.002 (0.000)	0.002 (0.000)	0.002 (-0.000)	0.001 (-0.000)

(b) $T = \frac{1}{4}$

K	Exact (0.000)	BS (0.041)	A (0.002)	BGM/L (0.002)	D (0.002)	JR (0.006)
70	30.005 (0.000)	30.001 (-0.004)	30.004 (-0.001)	30.004 (-0.001)	30.004 (-0.002)	30.006 (0.001)
80	20.104 (0.000)	20.065 (-0.039)	20.105 (0.001)	20.106 (0.002)	20.103 (-0.001)	20.113 (0.009)
85	15.341 (0.000)	15.278 (-0.063)	15.346 (0.005)	15.345 (0.004)	15.342 (0.001)	15.341 (0.000)
90	10.931 (0.000)	10.874 (-0.058)	10.935 (0.003)	10.933 (0.001)	10.932 (0.001)	10.915 (-0.017)
100	4.218 (0.000)	4.273 (0.054)	4.218 (-0.001)	4.218 (-0.001)	4.222 (0.003)	4.214 (-0.005)
105	2.236 (0.000)	2.325 (0.089)	2.236 (-0.000)	2.235 (-0.001)	2.241 (0.005)	2.248 (0.012)
115	0.465 (0.000)	0.518 (0.053)	0.463 (-0.002)	0.461 (-0.004)	0.464 (-0.001)	0.473 (0.008)
125	0.072 (0.000)	0.081 (0.010)	0.071 (-0.001)	0.072 (0.001)	0.071 (-0.001)	0.069 (-0.002)
135	0.009 (0.000)	0.009 (0.000)	0.010 (0.000)	0.011 (0.002)	0.009 (0.000)	0.008 (-0.002)

(c) $T = \frac{1}{2}$

K	Exact (0.000)	BS (0.067)	A (0.003)	BGM/L (0.004)	D (0.003)	JR (0.016)
60	40.010 (0.000)	40.002 (-0.008)	40.007 (-0.003)	40.008 (-0.002)	40.007 (-0.003)	40.015 (0.005)
70	30.101 (0.000)	30.053 (-0.048)	30.102 (0.000)	30.103 (0.002)	30.099 (-0.002)	30.122 (0.020)
80	20.596 (0.000)	20.497 (-0.099)	20.606 (0.009)	20.604 (0.007)	20.600 (0.004)	20.591 (-0.006)
90	12.291 (0.000)	12.247 (-0.044)	12.294 (0.003)	12.291 (0.000)	12.294 (0.002)	12.245 (-0.046)
100	6.182 (0.000)	6.279 (0.096)	6.180 (-0.002)	6.180 (-0.003)	6.187 (0.005)	6.168 (-0.014)
110	2.606 (0.000)	2.760 (0.154)	2.603 (-0.003)	2.601 (-0.005)	2.611 (0.005)	2.632 (0.026)
120	0.947 (0.000)	1.054 (0.107)	0.942 (-0.005)	0.939 (-0.008)	0.946 (-0.001)	0.969 (0.022)
130	0.310 (0.000)	0.356 (0.046)	0.307 (-0.004)	0.307 (-0.003)	0.307 (-0.003)	0.315 (0.005)
150	0.029 (0.000)	0.031 (0.002)	0.030 (0.001)	0.033 (0.004)	0.029 (-0.000)	0.025 (-0.004)

(d) $T = 1$

K	Exact (0.000)	BS (0.101)	A (0.005)	BGM/L (0.006)	D (0.004)	JR (0.018)
50	50.023 (0.000)	50.006 (-0.017)	50.019 (-0.004)	50.021 (-0.002)	50.019 (-0.005)	50.022 (-0.002)
60	40.151 (0.000)	40.086 (-0.065)	40.152 (0.001)	40.154 (0.003)	40.148 (-0.003)	40.176 (0.026)
70	30.638 (0.000)	30.516 (-0.122)	30.650 (0.012)	30.648 (0.010)	30.643 (0.005)	30.673 (0.035)
80	21.952 (0.000)	21.843 (-0.109)	21.963 (0.011)	21.958 (0.006)	21.958 (0.006)	21.940 (-0.012)
100	9.084 (0.000)	9.221 (0.138)	9.080 (-0.003)	9.078 (-0.005)	9.090 (0.007)	9.040 (-0.043)
110	5.268 (0.000)	5.486 (0.217)	5.264 (-0.005)	5.261 (-0.008)	5.276 (0.007)	5.261 (-0.007)
130	1.519 (0.000)	1.691 (0.172)	1.512 (-0.008)	1.508 (-0.011)	1.518 (-0.001)	1.543 (0.024)
150	0.390 (0.000)	0.456 (0.066)	0.385 (-0.005)	0.388 (-0.002)	0.386 (-0.004)	0.399 (0.009)
180	0.049 (0.000)	0.055 (0.006)	0.049 (0.000)	0.055 (0.006)	0.048 (-0.001)	0.047 (-0.002)

Table 5.7.: Exact and approximated option prices with errors for different strikes K and maturities T in the Heston model for parameters as in Table 5.3 (medium correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò12], **BGM/L** to the approximation by [BGM10b] resp. [Lew01], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = 2$

K	Exact (0.000)	BS (0.123)	A (0.005)	BGM/L (0.006)	D (0.005)	JR (0.154)
40	60.048 (0.000)	60.021 (-0.027)	60.044 (-0.003)	60.046 (-0.001)	60.043 (-0.004)	59.712 (-0.336)
50	50.250 (0.000)	50.170 (-0.079)	50.252 (0.002)	50.254 (0.004)	50.248 (-0.002)	50.267 (0.017)
60	40.858 (0.000)	40.727 (-0.131)	40.868 (0.010)	40.866 (0.008)	40.861 (0.003)	41.276 (0.418)
80	24.607 (0.000)	24.544 (-0.063)	24.614 (0.007)	24.609 (0.002)	24.613 (0.006)	24.854 (0.247)
100	13.205 (0.000)	13.368 (0.163)	13.203 (-0.002)	13.200 (-0.005)	13.214 (0.009)	13.052 (-0.153)
120	6.495 (0.000)	6.783 (0.288)	6.490 (-0.005)	6.486 (-0.009)	6.505 (0.009)	6.329 (-0.166)
150	2.043 (0.000)	2.271 (0.229)	2.034 (-0.009)	2.032 (-0.011)	2.042 (-0.000)	2.019 (-0.024)
180	0.620 (0.000)	0.730 (0.111)	0.612 (-0.007)	0.617 (-0.002)	0.615 (-0.005)	0.639 (0.019)
230	0.089 (0.000)	0.110 (0.020)	0.088 (-0.001)	0.096 (0.007)	0.087 (-0.002)	0.098 (0.009)

(b) $T = 4$

K	Exact (0.000)	BS (0.148)	A (0.005)	BGM/L (0.004)	D (0.005)	JR (2.515)
20	80.007 (0.000)	80.002 (-0.005)	80.006 (-0.001)	80.006 (-0.001)	80.006 (-0.001)	73.648 (-6.359)
30	70.085 (0.000)	70.053 (-0.032)	70.084 (-0.001)	70.085 (-0.000)	70.082 (-0.003)	66.699 (-3.386)
50	51.257 (0.000)	51.135 (-0.122)	51.264 (0.007)	51.262 (0.005)	51.258 (0.001)	56.820 (5.563)
60	42.861 (0.000)	42.739 (-0.122)	42.869 (0.008)	42.865 (0.004)	42.864 (0.003)	47.621 (4.760)
100	18.908 (0.000)	19.085 (0.177)	18.907 (-0.001)	18.905 (-0.003)	18.918 (0.010)	18.024 (-0.884)
120	12.135 (0.000)	12.438 (0.303)	12.133 (-0.003)	12.130 (-0.005)	12.148 (0.013)	10.999 (-1.136)
160	4.920 (0.000)	5.261 (0.341)	4.914 (-0.006)	4.911 (-0.008)	4.928 (0.008)	4.432 (-0.487)
230	1.062 (0.000)	1.235 (0.173)	1.054 (-0.009)	1.060 (-0.003)	1.059 (-0.004)	1.081 (0.019)
310	0.214 (0.000)	0.268 (0.055)	0.209 (-0.005)	0.218 (0.005)	0.209 (-0.004)	0.253 (0.040)

(c) $T = 8$

K	Exact (0.000)	BS (0.168)	A (0.003)	BGM/L (0.002)	D (0.005)	JR (62.224)
10	90.005 (0.000)	90.002 (-0.002)	90.004 (-0.000)	90.004 (-0.000)	90.004 (-0.001)	-192.43 (-282.4)
20	80.125 (0.000)	80.096 (-0.029)	80.125 (-0.000)	80.125 (0.000)	80.123 (-0.002)	139.08 (58.96)
30	70.655 (0.000)	70.583 (-0.072)	70.658 (0.002)	70.657 (0.002)	70.654 (-0.001)	210.13 (139.5)
50	53.942 (0.000)	53.848 (-0.094)	53.947 (0.005)	53.944 (0.002)	53.943 (0.001)	108.99 (55.05)
100	26.720 (0.000)	26.915 (0.195)	26.720 (-0.001)	26.718 (-0.002)	26.730 (0.010)	15.742 (-10.98)
120	20.295 (0.000)	20.597 (0.301)	20.294 (-0.001)	20.292 (-0.003)	20.309 (0.013)	10.428 (-9.867)
190	8.221 (0.000)	8.641 (0.420)	8.216 (-0.004)	8.215 (-0.005)	8.234 (0.013)	5.266 (-2.955)
300	2.387 (0.000)	2.671 (0.285)	2.377 (-0.009)	2.383 (-0.003)	2.387 (0.001)	2.184 (-0.203)
470	0.493 (0.000)	0.603 (0.110)	0.485 (-0.008)	0.496 (0.003)	0.488 (-0.005)	0.586 (0.093)

(d) $T = 16$

K	Exact (0.000)	BS (0.192)	A (0.004)	BGM/L (0.001)	D (0.005)	JR (6666.4)
5	95.009 (0.000)	95.006 (-0.003)	95.008 (-0.000)	95.009 (-0.000)	95.008 (-0.000)	7105 (7010)
10	90.103 (0.000)	90.086 (-0.017)	90.103 (-0.000)	90.103 (0.000)	90.102 (-0.001)	36343 (36253)
20	80.880 (0.000)	80.828 (-0.052)	80.882 (0.002)	80.881 (0.001)	80.879 (-0.001)	13013 (12933)
30	72.652 (0.000)	72.585 (-0.067)	72.654 (0.002)	72.653 (0.001)	72.651 (-0.000)	3383 (3310)
100	37.236 (0.000)	37.459 (0.223)	37.235 (-0.001)	37.235 (-0.001)	37.246 (0.010)	-400.38 (-437.6)
220	15.511 (0.000)	16.016 (0.505)	15.507 (-0.003)	15.508 (-0.003)	15.529 (0.018)	-33.916 (-49.43)
410	5.582 (0.000)	6.025 (0.443)	5.574 (-0.008)	5.579 (-0.003)	5.591 (0.009)	1.548 (-4.034)
770	1.411 (0.000)	1.636 (0.225)	1.400 (-0.011)	1.411 (0.001)	1.408 (-0.003)	1.484 (0.073)
850	1.100 (0.000)	1.294 (0.194)	1.090 (-0.010)	1.101 (0.001)	1.096 (-0.004)	1.228 (0.128)

Table 5.8.: Exact and approximated option prices with errors for different strikes K and maturities T in the Heston model for parameters as in Table 5.3 (medium correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò12], **BGM/L** to the approximation by [BGM10b] resp. [Lew01], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.029)	A (0.002)	BGM/L (0.001)	D (0.007)	JR (0.003)
80	20.001 (0.000)	20.000 (-0.001)	20.001 (-0.001)	20.001 (-0.000)	20.000 (-0.001)	20.001 (-0.000)
85	15.016 (0.000)	15.005 (-0.011)	15.014 (-0.002)	15.016 (-0.000)	15.011 (-0.005)	15.016 (-0.000)
90	10.129 (0.000)	10.086 (-0.043)	10.130 (0.002)	10.130 (0.001)	10.118 (-0.011)	10.133 (0.004)
95	5.678 (0.000)	5.621 (-0.058)	5.681 (0.003)	5.678 (0.000)	5.667 (-0.012)	5.676 (-0.002)
100	2.351 (0.000)	2.368 (0.017)	2.351 (-0.001)	2.351 (0.000)	2.358 (0.007)	2.344 (-0.007)
105	0.627 (0.000)	0.704 (0.076)	0.630 (0.002)	0.627 (-0.000)	0.649 (0.022)	0.632 (0.004)
110	0.097 (0.000)	0.143 (0.046)	0.096 (-0.001)	0.095 (-0.002)	0.107 (0.010)	0.100 (0.003)
115	0.008 (0.000)	0.020 (0.011)	0.006 (-0.002)	0.009 (0.000)	0.009 (0.000)	0.008 (-0.001)
120	0.000 (0.000)	0.002 (0.001)	-0.000 (-0.001)	0.001 (0.001)	0.000 (-0.000)	-0.000 (-0.001)

(b) $T = \frac{1}{4}$

K	Exact (0.000)	BS (0.090)	A (0.006)	BGM/L (0.004)	D (0.022)	JR (0.009)
70	30.010 (0.000)	30.001 (-0.009)	30.006 (-0.005)	30.008 (-0.002)	30.004 (-0.006)	30.008 (-0.003)
80	20.141 (0.000)	20.065 (-0.076)	20.141 (-0.000)	20.146 (0.005)	20.119 (-0.022)	20.152 (0.011)
85	15.407 (0.000)	15.278 (-0.128)	15.419 (0.012)	15.413 (0.007)	15.380 (-0.026)	15.424 (0.017)
90	11.013 (0.000)	10.874 (-0.140)	11.028 (0.015)	11.016 (0.003)	10.989 (-0.024)	11.013 (0.000)
100	4.209 (0.000)	4.273 (0.064)	4.209 (-0.000)	4.209 (-0.000)	4.234 (0.025)	4.183 (-0.025)
105	2.151 (0.000)	2.325 (0.175)	2.155 (0.004)	2.150 (-0.001)	2.205 (0.054)	2.149 (-0.002)
115	0.358 (0.000)	0.518 (0.160)	0.356 (-0.002)	0.347 (-0.010)	0.394 (0.036)	0.373 (0.015)
125	0.034 (0.000)	0.081 (0.047)	0.024 (-0.010)	0.032 (-0.002)	0.034 (0.000)	0.033 (-0.001)
135	0.002 (0.000)	0.009 (0.007)	-0.002 (-0.004)	0.007 (0.005)	-0.000 (-0.003)	0.000 (-0.002)

(c) $T = \frac{1}{2}$

K	Exact (0.000)	BS (0.157)	A (0.011)	BGM/L (0.009)	D (0.037)	JR (0.019)
60	40.020 (0.000)	40.002 (-0.018)	40.011 (-0.009)	40.016 (-0.004)	40.008 (-0.012)	40.007 (-0.013)
70	30.146 (0.000)	30.053 (-0.092)	30.141 (-0.005)	30.152 (0.006)	30.115 (-0.031)	30.150 (0.004)
80	20.701 (0.000)	20.497 (-0.204)	20.727 (0.026)	20.715 (0.013)	20.664 (-0.037)	20.751 (0.050)
90	12.399 (0.000)	12.247 (-0.152)	12.418 (0.019)	12.401 (0.002)	12.382 (-0.017)	12.392 (-0.007)
100	6.157 (0.000)	6.279 (0.122)	6.159 (0.002)	6.155 (-0.002)	6.204 (0.047)	6.103 (-0.053)
110	2.431 (0.000)	2.760 (0.329)	2.438 (0.007)	2.422 (-0.008)	2.527 (0.097)	2.423 (-0.007)
120	0.753 (0.000)	1.054 (0.301)	0.750 (-0.003)	0.733 (-0.021)	0.823 (0.070)	0.775 (0.022)
130	0.188 (0.000)	0.356 (0.168)	0.169 (-0.019)	0.172 (-0.016)	0.208 (0.020)	0.200 (0.012)
150	0.008 (0.000)	0.031 (0.023)	-0.003 (-0.011)	0.018 (0.011)	0.001 (-0.006)	0.005 (-0.002)

(d) $T = 1$

K	Exact (0.000)	BS (0.227)	A (0.018)	BGM/L (0.014)	D (0.054)	JR (0.111)
50	50.042 (0.000)	50.006 (-0.036)	50.029 (-0.013)	50.038 (-0.004)	50.022 (-0.020)	49.907 (-0.135)
60	40.213 (0.000)	40.086 (-0.127)	40.208 (-0.005)	40.222 (0.009)	40.172 (-0.041)	40.157 (-0.056)
70	30.764 (0.000)	30.516 (-0.248)	30.794 (0.029)	30.783 (0.019)	30.717 (-0.047)	30.979 (0.215)
80	22.110 (0.000)	21.843 (-0.267)	22.151 (0.041)	22.121 (0.012)	22.075 (-0.035)	22.374 (0.264)
100	9.028 (0.000)	9.221 (0.193)	9.035 (0.006)	9.022 (-0.006)	9.102 (0.074)	8.896 (-0.132)
110	5.056 (0.000)	5.486 (0.430)	5.063 (0.007)	5.044 (-0.012)	5.185 (0.129)	4.903 (-0.154)
130	1.225 (0.000)	1.691 (0.467)	1.216 (-0.009)	1.194 (-0.031)	1.331 (0.106)	1.209 (-0.016)
150	0.225 (0.000)	0.456 (0.231)	0.190 (-0.035)	0.208 (-0.017)	0.246 (0.021)	0.247 (0.023)
180	0.013 (0.000)	0.055 (0.042)	-0.007 (-0.020)	0.030 (0.017)	0.003 (-0.011)	0.020 (0.006)

Table 5.9.: Exact and approximated option prices with errors for different strikes K and maturities T in the Heston model for parameters as in Table 5.4 (high correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò12], **BGM/L** to the approximation by [BGM10b] resp. [Lew01], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = 2$

K	Exact (0.000)	BS (0.291)	A (0.022)	BGM/L (0.014)	D (0.066)	JR (0.601)
40	60.077 (0.000)	60.021 (-0.056)	60.064 (-0.013)	60.076 (-0.001)	60.051 (-0.026)	58.711 (-1.366)
50	50.330 (0.000)	50.170 (-0.159)	50.330 (0.000)	50.340 (0.010)	50.285 (-0.045)	50.304 (-0.026)
60	41.000 (0.000)	40.727 (-0.273)	41.029 (0.029)	41.017 (0.017)	40.947 (-0.053)	42.548 (1.548)
80	24.750 (0.000)	24.544 (-0.206)	24.787 (0.036)	24.755 (0.005)	24.735 (-0.016)	25.874 (1.123)
100	13.111 (0.000)	13.368 (0.258)	13.121 (0.010)	13.103 (-0.008)	13.203 (0.093)	12.662 (-0.448)
120	6.157 (0.000)	6.783 (0.626)	6.163 (0.007)	6.138 (-0.019)	6.332 (0.176)	5.499 (-0.657)
150	1.657 (0.000)	2.271 (0.615)	1.638 (-0.019)	1.625 (-0.031)	1.791 (0.134)	1.496 (-0.160)
180	0.387 (0.000)	0.730 (0.343)	0.340 (-0.047)	0.371 (-0.016)	0.426 (0.039)	0.432 (0.045)
230	0.030 (0.000)	0.110 (0.079)	-0.004 (-0.034)	0.048 (0.018)	0.017 (-0.013)	0.067 (0.036)

(b) $T = 4$

K	Exact (0.000)	BS (0.344)	A (0.021)	BGM/L (0.009)	D (0.078)	JR (7.257)
20	80.013 (0.000)	80.002 (-0.011)	80.009 (-0.005)	80.012 (-0.002)	80.007 (-0.006)	60.724 (-19.29)
30	70.120 (0.000)	70.053 (-0.067)	70.113 (-0.007)	70.121 (0.001)	70.096 (-0.024)	61.256 (-8.864)
50	51.400 (0.000)	51.135 (-0.264)	51.424 (0.025)	51.410 (0.010)	51.348 (-0.052)	67.655 (16.26)
60	43.027 (0.000)	42.739 (-0.288)	43.061 (0.034)	43.035 (0.008)	42.980 (-0.047)	56.604 (13.58)
100	18.769 (0.000)	19.085 (0.316)	18.779 (0.009)	18.763 (-0.006)	18.873 (0.103)	16.269 (-2.501)
120	11.791 (0.000)	12.438 (0.647)	11.797 (0.006)	11.780 (-0.011)	11.973 (0.182)	8.551 (-3.239)
160	4.412 (0.000)	5.261 (0.849)	4.405 (-0.007)	4.391 (-0.021)	4.621 (0.209)	2.978 (-1.434)
230	0.746 (0.000)	1.235 (0.489)	0.695 (-0.051)	0.733 (-0.013)	0.819 (0.073)	0.787 (0.040)
310	0.102 (0.000)	0.268 (0.166)	0.053 (-0.049)	0.113 (0.011)	0.099 (-0.003)	0.215 (0.113)

(c) $T = 8$

K	Exact (0.000)	BS (0.398)	A (0.017)	BGM/L (0.006)	D (0.090)	JR (167.19)
10	90.008 (0.000)	90.002 (-0.006)	90.006 (-0.002)	90.007 (-0.001)	90.005 (-0.003)	-646.74 (-736.8)
20	80.158 (0.000)	80.096 (-0.063)	80.156 (-0.003)	80.159 (0.001)	80.139 (-0.019)	279.43 (199.3)
30	70.740 (0.000)	70.583 (-0.157)	70.748 (0.008)	70.744 (0.004)	70.704 (-0.036)	442.24 (371.5)
50	54.077 (0.000)	53.848 (-0.229)	54.100 (0.022)	54.081 (0.003)	54.037 (-0.040)	188.94 (134.9)
100	26.529 (0.000)	26.915 (0.386)	26.534 (0.004)	26.525 (-0.004)	26.643 (0.113)	-2.135 (-28.66)
120	19.943 (0.000)	20.597 (0.653)	19.946 (0.002)	19.937 (-0.006)	20.122 (0.178)	-5.375 (-25.32)
190	7.608 (0.000)	8.641 (1.032)	7.601 (-0.008)	7.595 (-0.013)	7.863 (0.255)	0.054 (-7.554)
300	1.917 (0.000)	2.671 (0.754)	1.870 (-0.047)	1.905 (-0.012)	2.062 (0.145)	1.391 (-0.526)
470	0.303 (0.000)	0.603 (0.300)	0.243 (-0.059)	0.309 (0.006)	0.325 (0.022)	0.547 (0.244)

(d) $T = 16$

K	Exact (0.000)	BS (0.470)	A (0.023)	BGM/L (0.003)	D (0.100)	JR (19176)
5	95.012 (0.000)	95.006 (-0.006)	95.011 (-0.001)	95.012 (-0.000)	95.010 (-0.002)	40575 (40480)
10	90.122 (0.000)	90.086 (-0.036)	90.122 (-0.001)	90.123 (0.000)	90.112 (-0.010)	93950 (93860)
20	80.945 (0.000)	80.828 (-0.117)	80.951 (0.006)	80.946 (0.002)	80.919 (-0.026)	30134 (30053)
30	72.742 (0.000)	72.585 (-0.157)	72.754 (0.012)	72.744 (0.002)	72.711 (-0.031)	7096 (7024)
100	36.983 (0.000)	37.459 (0.476)	36.981 (-0.001)	36.980 (-0.003)	37.111 (0.128)	-1012 (-1049)
220	14.795 (0.000)	16.016 (1.221)	14.786 (-0.009)	14.787 (-0.007)	15.099 (0.304)	-101.92 (-116.7)
410	4.898 (0.000)	6.025 (1.126)	4.857 (-0.041)	4.889 (-0.009)	5.144 (0.246)	-4.697 (-9.595)
770	1.051 (0.000)	1.636 (0.585)	0.982 (-0.070)	1.052 (0.001)	1.137 (0.086)	1.251 (0.200)
850	0.792 (0.000)	1.294 (0.502)	0.722 (-0.070)	0.794 (0.003)	0.857 (0.066)	1.122 (0.331)

Table 5.10.: Exact and approximated option prices with errors for different strikes K and maturities T in the Heston model for parameters as in Table 5.4 (high correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò12], **BGM/L** to the approximation by [BGM10b] resp. [Lew01], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

5.5.3. Extended Stein & Stein model

5.5.3.1. Model specification

In the Stein & Stein model [SS91], extended by correlation between stock and volatility by [SZ99], the logarithm $X = \log(S)$ of the discounted stock price process is given by

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}y_t\,\mathrm{d}t + y_t\left(\rho\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2\right), \quad X_0 = \log(S_0), \\ \mathrm{d}y_t &= \kappa(\eta - y_t)\,\mathrm{d}t + \theta\,\mathrm{d}W_t^1, \quad y_0 > 0, \end{aligned} \tag{5.38}$$

for independent standard Brownian motions W^1, W^2, $\rho \in [-1,1]$, κ, η, $\theta > 0$.

The following lemma provides the regularity condition required to apply Proposition 5.3.2, which allows us to interpret the model from (5.38) as stochastic volatility model in the sense of Definition 5.2.1.

Lemma 5.5.10. *In the extended Stein & Stein model* (5.38)*, we have* $\int_0^\infty y_t^2\,\mathrm{d}t = \infty$.

PROOF. Cf. Section 5.7.5. □

Hence, for two independent standard Brownian motions W^1, W^2, parameters $\rho \in [-1,1]$, κ, η, $\theta > 0$, and the process y as in (5.38), we can consider the extended Stein & Stein model to be generated by a stochastic volatility model (S_0, L, V, U) in the sense of Definition 5.2.1 with

$$\begin{aligned} L_t &= \sqrt{1-\rho^2}W_t^2 - \frac{1}{2}(1-\rho^2)t, \\ V_t &= \int_0^t y_s^2\,\mathrm{d}s, \\ M_t &= \rho\int_0^t y_s\,\mathrm{d}W_s^1, \\ U_t &= M_t - \frac{1}{2}\rho^2\int_0^t y_s^2\,\mathrm{d}s \end{aligned} \tag{5.39}$$

for $t \in \mathbb{R}_+$.

It is part of Definition 5.2.1 that the process V is almost surely strictly increasing. By the next lemma, this is indeed the case for the extended Stein & Stein model.

Lemma 5.5.11. *In the extended Stein & Stein model* (5.39)*, the mapping* $t \mapsto \int_0^t y_s^2\,\mathrm{d}s$ *is almost surely strictly increasing.*

PROOF. Cf. Section 5.7.5. □

5.5.3.2. Check of regularity conditions

We now verify that the assumptions necessary for our analysis (cf. Table 5.1) hold in general in the extended Stein & Stein model, or we provide sufficient criteria on the model parameters.

Since L_1 is normally distributed, Assumptions 5.2.4 and 5.2.21(2) obviously hold. Assumption 5.2.9 clearly holds as well. Finally, Assumptions 5.2.16 and 5.2.26 are no restrictions since M is continuous.

Lemma 5.5.12 (Assumption 5.2.8)**.** *In the extended Stein & Stein model* (5.39)*, M is a martingale.*

PROOF. For all $t \in \mathbb{R}_+$, we have $\mathrm{E}\left(\rho^2 \int_0^t y_s^2\, ds\right) = \rho^2 \mathrm{E}(V_t) < \infty$ by Lemma 5.5.17 below, which implies that the local martingale M is a martingale. □

Lemma 5.5.13 (Criterion for Assumption 5.2.11)**.** *In the Stein & Stein model* (5.39)*, if $\rho^2 < \frac{\kappa^2}{\theta^2}$, then the process e^{U^λ} defined in* (5.12) *is a martingale for all $\lambda \in [0,1]$.*

PROOF. Cf. Section 5.7.5. □

Lemma 5.5.14 (Assumption 5.2.21(1a))**.** *In the extended Stein & Stein model* (5.39)*, for all $t \in \mathbb{R}_+$, there exist b_1, $b_2 > 0$ such that*

$$\mathrm{E}\left(e^{uV_t}\right) \leq e^{b_1 - b_2\sqrt{|u|}} \quad \textit{for all } u \in \mathbb{R}_-.$$

PROOF. Cf. Section 5.7.5. □

The next lemma provides a criterion to check Assumption 5.2.21(3), which depends also on $\kappa^\lambda(R)$. Note that since L is a Brownian motion with drift in the extended Stein & Stein model, we have by construction

$$\kappa^\lambda(z) = \kappa(z) = \frac{1}{2}(1-\rho^2)z(z-1) \quad \text{for all } \lambda \in [0,1],\ z \in \mathbb{C}. \tag{5.40}$$

Lemma 5.5.15 (Criterion for Assumption 5.2.21(3))**.** *In the extended Stein & Stein model* (5.39)*, for all $t \in \mathbb{R}_+$ and for all $a < \frac{\kappa^2}{2\theta^2}$, it holds $\mathrm{E}\left(e^{aV_t}\right) < \infty$.*

PROOF. Cf. Lemma 5.7.18 in Section 5.7.5. □

Lemma 5.5.16 (Criterion for Assumption 5.2.21(4))**.** *In the extended Stein & Stein model* (5.39)*, for all $t \in \mathbb{R}_+$ and all $a \in \mathbb{R}$ with $|a\rho| < \frac{\kappa}{2\theta}$, it holds*

$$\mathrm{E}\left(e^{aM_t}\right) < \infty.$$

Moreover, for all $t \in \mathbb{R}_+$ and all $b \in \mathbb{R}$ such that $|\rho b| < \frac{\kappa}{4\theta}$ and $-b\rho^2 < \frac{\kappa^2}{2\theta^2}$, we have

$$\mathrm{E}\left(e^{bU_t}\right) < \infty.$$

PROOF. Cf. Section 5.7.5. □

Lemma 5.5.17 (Assumptions 5.2.15 and 5.2.25)**.** *In the extended Stein & Stein model* (5.39)*, for all $t \in \mathbb{R}_+$, the random variables M_t, $\langle M,M\rangle_t$, and V_t have moments of any order.*

PROOF. By Lemma 5.5.16, there is $\varepsilon > 0$ such that for all $t \in \mathbb{R}_+$, it holds $\mathrm{E}\left(e^{\pm\varepsilon M_t}\right) < \infty$. This implies the existence of all moments of M_t. By Lemma 5.5.15, we can apply the same argument to $\langle M,M\rangle_t = \rho^2 V_t$ and V_t, $t \in \mathbb{R}_+$. □

5.5.3.3. Moments required in the approximation

In the case of the extended Stein & Stein model, the moments used in the approximate option pricing formula from Theorem 5.4.3 are given by

$$
\begin{aligned}
\operatorname{Var}(L_1) &= 1-\rho^2,\\
\operatorname{Skew}(L_1) &= 0,\\
\operatorname{ExKurt}(L_1) &= 0,\\
\mathrm{E}(V_T) &= -\frac{1}{4\kappa^2}\Big(\theta^2-4y_0\eta\kappa-2{y_0}^2\kappa+6\kappa\eta^2-2\theta^2T\kappa-\theta^2e^{-2\kappa T}\\
&\quad -4e^{-2\kappa T}y_0\eta\kappa+2e^{-2\kappa T}{y_0}^2\kappa+2e^{-2\kappa T}\eta^2\kappa-8\eta^2e^{-\kappa T}\kappa\\
&\quad +8y_0e^{-\kappa T}\eta\kappa-4\eta^2\kappa^2T\Big),\\
\operatorname{Var}(V_T) &= \frac{\theta^2}{8\kappa^4}e^{-4\kappa T}\Big(-5e^{4\kappa T}\theta^2+24e^{4\kappa T}y_0\eta\kappa-76e^{4\kappa T}\kappa\eta^2+4e^{4\kappa T}{y_0}^2\kappa\\
&\quad +4\theta^2e^{2\kappa T}+32y_0\eta e^{2\kappa T}\kappa-48y_0\eta e^{3\kappa T}\kappa-16y_0\eta e^{\kappa T}\kappa-32y_0\eta e^{3\kappa T}\kappa^2T\\
&\quad +32y_0\eta e^{2\kappa T}\kappa^2T-4\kappa\eta^2+32\kappa^2\eta^2Te^{4\kappa T}+4\theta^2Te^{4\kappa T}\kappa+112\eta^2e^{3\kappa T}\kappa\\
&\quad -48\eta^2e^{2\kappa T}\kappa+16\eta^2e^{\kappa T}\kappa+32\eta^2e^{3\kappa T}\kappa^2T+8\theta^2e^{2\kappa T}\kappa T-16{y_0}^2e^{2\kappa T}\kappa^2T\\
&\quad -16\eta^2e^{2\kappa T}\kappa^2T+\theta^2-4{y_0}^2\kappa+8y_0\eta\kappa\Big),\\
\operatorname{Var}(M_T) &= \rho^2\mathrm{E}(V_T),\\
\operatorname{Cov}(V_T,M_T) &= \frac{\rho}{2\theta}\left(2\kappa\operatorname{Var}(V_T)+\int_0^T\operatorname{Cov}\left(y_T^2,y_t^2\right)\,\mathrm{d}t-2\kappa\eta\int_0^T\int_0^T\operatorname{Cov}\left(y_s,y_t^2\right)\,\mathrm{d}s\,\mathrm{d}t\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^T\operatorname{Cov}\left(y_T^2,y_t^2\right)\,\mathrm{d}t &= \frac{\theta^2}{4\kappa^3}e^{-4\kappa T}\Big(-20\eta^2e^{3\kappa T}\kappa-4\kappa e^{2\kappa T}{y_0}^2-8y_0\eta e^{2\kappa T}\kappa\\
&\quad +20\eta^2e^{2\kappa T}\kappa+4y_0\eta e^{3\kappa T}\kappa+12y_0\eta e^{\kappa T}\kappa-8y_0\eta\kappa\\
&\quad -16y_0\eta e^{2\kappa T}\kappa^2T+8y_0\eta e^{3\kappa T}\kappa^2T-12\eta^2e^{\kappa T}\kappa\\
&\quad +8\eta^2e^{2\kappa T}\kappa^2T+4\kappa\eta^2+e^{4\kappa T}\theta^2+8e^{4\kappa T}\kappa\eta^2\\
&\quad +4{y_0}^2\kappa-\theta^2+8{y_0}^2e^{2\kappa T}\kappa^2T-4\theta^2e^{2\kappa T}\kappa T-8\eta^2e^{3\kappa T}\kappa^2T\Big),\\
\int_0^T\int_0^T\operatorname{Cov}\left(y_s,y_t^2\right)\,\mathrm{d}s\,\mathrm{d}t &= -\frac{\theta^2}{2\kappa^3}e^{-3\kappa T}\Big(-2e^{3\kappa T}y_0+8\eta e^{3\kappa T}-11\eta e^{2\kappa T}\\
&\quad +3y_0e^{2\kappa T}-2y_0e^{\kappa T}+4\eta e^{\kappa T}+2y_0T\kappa e^{2\kappa T}+y_0\\
&\quad -\eta-4\eta T\kappa e^{3\kappa T}-2\eta T\kappa e^{2\kappa T}\Big).
\end{aligned}
$$

The derivation is delegated to Appendix B.2.

S_0	r	y_0	κ	η	θ	ρ
100	0.0953	0.2	4.00	0.2	0.1	0

Table 5.11.: Extended Stein & Stein model parameters (zero correlation)

S_0	r	y_0	κ	η	θ	ρ
100	0.0953	0.2	4.00	0.2	0.1	-0.5

Table 5.12.: Extended Stein & Stein model parameters (high correlation)

5.5.3.4. Numerical comparison

We assess the quality of our approximation in the extended Stein & Stein model for three different parameter sets given in Tables 5.11, 5.12, and 5.13. Besides our generic benchmarks BS and JR (cf. Section 5.5.1.2), we compare our approximation to those of [Alò06], indicated by A, and [FPS00], indicated by FPS. We discuss these approaches in Sections 5.6.4 and 5.6.3.

Up to the correlation parameter ρ, all three parameter sets are taken from [SS91], and the first two are also considered in [Alò06]. They differ only by the choice of the correlation parameter ρ, where we set zero correlation ($\rho = 0$) in the first and high correlation ($\rho = -0.5$) in the second case. The third parameter set from Table 5.13 exhibits a medium correlation of $\rho = -0.2$, but a significantly higher volatility of volatility $\theta = 0.6$, compared to $\theta = 0.1$ in the first two cases.

In order to determine reasonable strikes for every maturity, we use the Black-Scholes volatility $\widetilde{\sigma} = \eta$, cf. Section 5.5.1.1. This corresponds to the long-term mean volatility in the extended Stein & Stein model, i.e., $\lim_{t\to\infty} \frac{1}{t}\int_0^t y_s \,\mathrm{d}s = \eta$. Note that η coincides for Tables 5.11 and 5.12, but it is different in Table 5.13.

We use the representation of the characteristic function of $X_T = \log(S_T)$ from [LK10] to compute the exact option price by the Laplace transform method (cf. Section 5.5.1.3).

As we see from Section 5.5.3.2, the only regularity conditions whose validity depends on the model parameters are Assumptions 5.2.11, 5.2.21(3), and 5.2.21(4). Since $\rho^2 < \frac{\kappa^2}{\theta^2}$ for all three parameter sets under consideration, Lemma 5.5.13 implies that Assumption 5.2.11 is satisfied. By the same argument as for the Heston model (cf. Section 5.5.2.4), Lemma 5.5.15 implies that

S_0	r	y_0	κ	η	θ	ρ
100	0.0953	0.35	16.00	0.35	0.6	-0.2

Table 5.13.: Extended Stein & Stein model parameters (high volatility of volatility)

Assumption 5.2.21(3) is always satisfied in the case of a European call option. The same reasoning and Lemma 5.5.16 show that Assumption 5.2.21(4) is satisfied in this case as well.

Tables 5.14 and 5.15 show the exact price, the different approximations, and their errors for parameters from Table 5.11, Tables 5.16 and 5.17 for parameters from Table 5.12, and Tables 5.18 and 5.19 for parameters from Table 5.13.

For parameters from Table 5.11, our approximation is always exact for the presented number of digits. In particular, we obtain a MAEOS of 0 for all maturities under consideration. We remark that approximation A coincides with the Black-Scholes price BS in the case $\rho = 0$. Moreover, note that the approximation FPS is outperformed by BS for all maturities up to $T = 16$, and JR is outperformed by BS for maturities $T = \frac{1}{2}$ and longer.

In the case of parameters from Table 5.12, our approximation exhibits a MAEOS of 0.003 for $T = \frac{1}{12}$, and it increases to a level of 0.030 for $T = 16$. In terms of the MAEOS, our approximation outperforms JR for maturities of $T = 1$ and longer, and it is superior to FPS up to the maturity $T = 1$. However, our approximation is outperformed for all maturities by the tailor-made approximation A. In the worst case, our MAEOS amounts to 0.030 for $T = 16$, while the MAEOS of A is given by 0.006. Hence, despite the generality of our approach, we obtain reasonable results even in comparison to a much more specific approximation.

In Table 5.13, we consider a medium correlation of $\rho = -0.2$, but a significantly increased volatility of volatility. The MAEOS of our approximation mostly stays on the level 0.004, for $T = \frac{1}{12}$ it amounts to 0.002, and for $T = 8$ to 0.005. In terms of the MAEOS, we clearly outperform all other approximations under consideration by at least one digit. As the experiments with parameters from Table 5.12 show, A and FPS are able to handle the case of high correlation quite well. Hence, the superiority of our approximation for parameters from Table 5.13 seems not to be due to the reduced correlation of $\rho = -0.2$, but due to the better capability to handle high volatility of volatility.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.001)	A (0.001)	D (0.000)	FPS (0.008)	JR (0.001)
84	16.666 (0.000)	16.666 (-0.000)	16.666 (-0.000)	16.666 (-0.000)	16.666 (-0.000)	16.666 (0.000)
88	12.715 (0.000)	12.714 (-0.002)	12.714 (-0.002)	12.715 (0.000)	12.715 (-0.000)	12.716 (0.001)
92	8.867 (0.000)	8.864 (-0.003)	8.864 (-0.003)	8.867 (0.000)	8.871 (0.005)	8.866 (-0.000)
96	5.390 (0.000)	5.392 (0.001)	5.392 (0.001)	5.390 (-0.000)	5.409 (0.019)	5.388 (-0.002)
100	2.715 (0.000)	2.721 (0.006)	2.721 (0.006)	2.715 (0.000)	2.746 (0.032)	2.714 (-0.001)
104	1.091 (0.000)	1.094 (0.004)	1.094 (0.004)	1.091 (-0.000)	1.117 (0.027)	1.092 (0.002)
108	0.345 (0.000)	0.343 (-0.001)	0.343 (-0.001)	0.345 (-0.000)	0.357 (0.012)	0.346 (0.001)
112	0.086 (0.000)	0.083 (-0.003)	0.083 (-0.003)	0.086 (0.000)	0.089 (0.002)	0.086 (-0.000)
116	0.017 (0.000)	0.016 (-0.002)	0.016 (-0.002)	0.018 (0.000)	0.017 (-0.000)	0.017 (-0.000)

(b) $T = \frac{1}{4}$

K	Exact (0.000)	BS (0.005)	A (0.005)	D (0.000)	FPS (0.010)	JR (0.004)
80	21.909 (0.000)	21.904 (-0.004)	21.904 (-0.004)	21.909 (0.000)	21.906 (-0.003)	21.914 (0.006)
85	17.125 (0.000)	17.118 (-0.008)	17.118 (-0.008)	17.126 (0.000)	17.122 (-0.004)	17.128 (0.002)
90	12.573 (0.000)	12.567 (-0.005)	12.567 (-0.005)	12.573 (-0.000)	12.578 (0.006)	12.564 (-0.008)
95	8.516 (0.000)	8.522 (0.007)	8.522 (0.007)	8.515 (-0.000)	8.542 (0.026)	8.501 (-0.015)
100	5.244 (0.000)	5.263 (0.019)	5.263 (0.019)	5.244 (0.000)	5.289 (0.044)	5.237 (-0.008)
110	1.463 (0.000)	1.471 (0.008)	1.471 (0.008)	1.463 (-0.000)	1.493 (0.030)	1.474 (0.011)
115	0.668 (0.000)	0.664 (-0.004)	0.664 (-0.004)	0.668 (-0.000)	0.679 (0.011)	0.676 (0.008)
125	0.110 (0.000)	0.101 (-0.009)	0.101 (-0.009)	0.110 (0.000)	0.105 (-0.005)	0.109 (-0.001)
130	0.041 (0.000)	0.034 (-0.006)	0.034 (-0.006)	0.041 (0.000)	0.036 (-0.004)	0.039 (-0.002)

(c) $T = \frac{1}{2}$

K	Exact (0.000)	BS (0.007)	A (0.007)	D (0.000)	FPS (0.010)	JR (0.009)
70	33.269 (0.000)	33.265 (-0.003)	33.265 (-0.003)	33.269 (0.000)	33.266 (-0.003)	33.282 (0.013)
80	23.875 (0.000)	23.863 (-0.012)	23.863 (-0.012)	23.875 (0.000)	23.866 (-0.009)	23.884 (0.009)
85	19.371 (0.000)	19.360 (-0.011)	19.360 (-0.011)	19.371 (0.000)	19.366 (-0.005)	19.359 (-0.013)
95	11.379 (0.000)	11.391 (0.013)	11.391 (0.013)	11.379 (-0.000)	11.407 (0.028)	11.340 (-0.039)
100	8.176 (0.000)	8.203 (0.027)	8.203 (0.027)	8.176 (-0.000)	8.222 (0.047)	8.150 (-0.026)
110	3.694 (0.000)	3.722 (0.028)	3.722 (0.028)	3.694 (-0.000)	3.743 (0.049)	3.710 (0.016)
125	0.837 (0.000)	0.828 (-0.009)	0.828 (-0.009)	0.837 (-0.000)	0.839 (0.002)	0.853 (0.016)
135	0.269 (0.000)	0.253 (-0.016)	0.253 (-0.016)	0.269 (0.000)	0.258 (-0.011)	0.270 (0.002)
150	0.043 (0.000)	0.034 (-0.009)	0.034 (-0.009)	0.043 (0.000)	0.035 (-0.008)	0.039 (-0.004)

(d) $T = 1$

K	Exact (0.000)	BS (0.009)	A (0.009)	D (0.000)	FPS (0.011)	JR (0.021)
65	40.937 (0.000)	40.932 (-0.006)	40.932 (-0.006)	40.937 (0.000)	40.932 (-0.005)	40.981 (0.044)
75	32.020 (0.000)	32.007 (-0.013)	32.007 (-0.013)	32.021 (0.000)	32.009 (-0.011)	32.038 (0.018)
85	23.594 (0.000)	23.585 (-0.009)	23.585 (-0.009)	23.594 (-0.000)	23.591 (-0.003)	23.536 (-0.058)
95	16.203 (0.000)	16.216 (0.013)	16.216 (0.013)	16.202 (-0.000)	16.227 (0.024)	16.117 (-0.086)
100	13.059 (0.000)	13.084 (0.026)	13.084 (0.026)	13.058 (-0.000)	13.097 (0.039)	12.989 (-0.070)
120	4.627 (0.000)	4.657 (0.030)	4.657 (0.030)	4.627 (-0.000)	4.672 (0.044)	4.664 (0.037)
140	1.315 (0.000)	1.306 (-0.009)	1.306 (-0.009)	1.314 (-0.000)	1.314 (-0.000)	1.345 (0.031)
160	0.327 (0.000)	0.307 (-0.020)	0.307 (-0.020)	0.327 (0.000)	0.311 (-0.016)	0.328 (0.001)
180	0.076 (0.000)	0.064 (-0.012)	0.064 (-0.012)	0.077 (0.000)	0.065 (-0.011)	0.071 (-0.006)

Table 5.14.: Exact and approximated option prices with errors for different strikes K and maturities T in the extended Stein & Stein model for parameters as in Table 5.11 (zero correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò06], **FPS** to the approximation by [FPS00], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = 2$

K	Exact (0.000)	BS (0.008)	A (0.008)	D (0.000)	FPS (0.010)	JR (0.045)
55	54.565 (0.000)	54.562 (-0.003)	54.562 (-0.003)	54.565 (0.000)	54.562 (-0.003)	54.687 (0.122)
65	46.401 (0.000)	46.392 (-0.009)	46.392 (-0.009)	46.401 (0.000)	46.393 (-0.008)	46.490 (0.089)
80	34.667 (0.000)	34.657 (-0.010)	34.657 (-0.010)	34.667 (-0.000)	34.660 (-0.007)	34.563 (-0.104)
94	24.927 (0.000)	24.931 (0.005)	24.931 (0.005)	24.926 (-0.000)	24.938 (0.011)	24.741 (-0.185)
115	13.745 (0.000)	13.777 (0.033)	13.777 (0.033)	13.745 (0.000)	13.788 (0.043)	13.688 (-0.056)
140	5.970 (0.000)	5.995 (0.025)	5.995 (0.025)	5.970 (-0.000)	6.005 (0.035)	6.038 (0.068)
170	1.962 (0.000)	1.954 (-0.008)	1.954 (-0.008)	1.962 (-0.000)	1.961 (-0.002)	2.009 (0.046)
200	0.609 (0.000)	0.590 (-0.019)	0.590 (-0.019)	0.610 (0.000)	0.593 (-0.016)	0.616 (0.007)
240	0.126 (0.000)	0.113 (-0.013)	0.113 (-0.013)	0.127 (0.000)	0.114 (-0.012)	0.120 (-0.007)

(b) $T = 4$

K	Exact (0.000)	BS (0.007)	A (0.007)	D (0.000)	FPS (0.008)	JR (0.159)
50	65.880 (0.000)	65.877 (-0.003)	65.877 (-0.003)	65.880 (0.000)	65.878 (-0.003)	66.387 (0.507)
60	59.149 (0.000)	59.144 (-0.006)	59.144 (-0.006)	59.149 (0.000)	59.144 (-0.005)	59.386 (0.237)
80	46.254 (0.000)	46.246 (-0.008)	46.246 (-0.008)	46.254 (0.000)	46.248 (-0.006)	45.844 (-0.410)
104	32.727 (0.000)	32.732 (0.005)	32.732 (0.005)	32.727 (-0.000)	32.737 (0.010)	32.248 (-0.479)
135	19.633 (0.000)	19.658 (0.025)	19.658 (0.025)	19.633 (0.000)	19.665 (0.032)	19.539 (-0.094)
170	10.446 (0.000)	10.470 (0.024)	10.470 (0.024)	10.446 (-0.000)	10.478 (0.032)	10.564 (0.118)
230	3.364 (0.000)	3.360 (-0.004)	3.360 (-0.004)	3.364 (-0.000)	3.365 (0.001)	3.444 (0.080)
290	1.087 (0.000)	1.071 (-0.016)	1.071 (-0.016)	1.087 (0.000)	1.074 (-0.014)	1.103 (0.015)
370	0.257 (0.000)	0.244 (-0.013)	0.244 (-0.013)	0.257 (0.000)	0.245 (-0.012)	0.251 (-0.006)

(c) $T = 8$

K	Exact (0.000)	BS (0.005)	A (0.005)	D (0.000)	FPS (0.006)	JR (1.492)
40	81.351 (0.000)	81.350 (-0.001)	81.350 (-0.001)	81.351 (0.000)	81.350 (-0.001)	86.534 (5.183)
60	72.147 (0.000)	72.144 (-0.003)	72.144 (-0.003)	72.148 (0.000)	72.144 (-0.003)	71.992 (-0.155)
90	59.048 (0.000)	59.043 (-0.005)	59.043 (-0.005)	59.048 (0.000)	59.044 (-0.004)	55.451 (-3.597)
126	45.330 (0.000)	45.332 (0.002)	45.332 (0.002)	45.330 (-0.000)	45.335 (0.005)	43.038 (-2.292)
180	29.709 (0.000)	29.726 (0.017)	29.726 (0.017)	29.709 (0.000)	29.731 (0.021)	29.511 (-0.198)
260	15.695 (0.000)	15.715 (0.020)	15.715 (0.020)	15.695 (0.000)	15.720 (0.025)	16.108 (0.413)
380	6.241 (0.000)	6.242 (0.001)	6.242 (0.001)	6.241 (-0.000)	6.247 (0.006)	6.439 (0.198)
540	2.025 (0.000)	2.013 (-0.013)	2.013 (-0.013)	2.025 (0.000)	2.015 (-0.010)	2.062 (0.037)
780	0.461 (0.000)	0.450 (-0.012)	0.450 (-0.012)	0.462 (0.000)	0.451 (-0.011)	0.457 (-0.004)

(d) $T = 16$

K	Exact (0.000)	BS (0.004)	A (0.004)	D (0.000)	FPS (0.004)	JR (95.767)
40	91.302 (0.000)	91.302 (-0.000)	91.302 (-0.000)	91.302 (-0.000)	91.302 (-0.000)	232.71 (141.4)
70	84.868 (0.000)	84.867 (-0.002)	84.867 (-0.002)	84.868 (0.000)	84.867 (-0.001)	-37.335 (-122.2)
120	74.684 (0.000)	74.681 (-0.003)	74.681 (-0.003)	74.684 (0.000)	74.681 (-0.002)	-40.334 (-115.0)
200	60.514 (0.000)	60.515 (0.001)	60.515 (0.001)	60.514 (-0.000)	60.516 (0.002)	30.437 (-30.08)
330	43.244 (0.000)	43.255 (0.011)	43.255 (0.011)	43.244 (-0.000)	43.257 (0.013)	47.185 (3.941)
560	25.001 (0.000)	25.017 (0.016)	25.017 (0.016)	25.001 (0.000)	25.021 (0.020)	29.306 (4.305)
930	11.697 (0.000)	11.703 (0.005)	11.703 (0.005)	11.697 (-0.000)	11.706 (0.009)	12.832 (1.135)
1500	4.481 (0.000)	4.473 (-0.008)	4.473 (-0.008)	4.481 (-0.000)	4.475 (-0.006)	4.658 (0.177)
2600	1.077 (0.000)	1.065 (-0.011)	1.065 (-0.011)	1.077 (0.000)	1.066 (-0.010)	1.080 (0.004)

Table 5.15.: Exact and approximated option prices with errors for different strikes K and maturities T in the extended Stein & Stein model for parameters as in Table 5.11 (zero correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò06], **FPS** to the approximation by [FPS00], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.012)	A (0.002)	D (0.003)	FPS (0.074)	JR (0.001)
84	16.668 (0.000)	16.666 (-0.003)	16.667 (-0.001)	16.667 (-0.001)	16.677 (0.009)	16.668 (-0.000)
88	12.726 (0.000)	12.714 (-0.013)	12.724 (-0.002)	12.722 (-0.004)	12.787 (0.061)	12.727 (0.000)
92	8.897 (0.000)	8.864 (-0.033)	8.896 (-0.001)	8.889 (-0.008)	9.085 (0.187)	8.900 (0.002)
96	5.430 (0.000)	5.392 (-0.039)	5.433 (0.003)	5.422 (-0.009)	5.681 (0.251)	5.430 (-0.000)
100	2.723 (0.000)	2.721 (-0.002)	2.729 (0.006)	2.723 (0.000)	2.798 (0.075)	2.719 (-0.004)
104	1.053 (0.000)	1.094 (0.042)	1.057 (0.005)	1.064 (0.012)	0.874 (-0.179)	1.052 (-0.001)
108	0.297 (0.000)	0.343 (0.046)	0.297 (-0.000)	0.309 (0.012)	0.050 (-0.247)	0.300 (0.002)
112	0.059 (0.000)	0.083 (0.024)	0.056 (-0.004)	0.064 (0.005)	-0.099 (-0.158)	0.060 (0.001)
116	0.008 (0.000)	0.016 (0.008)	0.005 (-0.003)	0.009 (0.001)	-0.055 (-0.063)	0.008 (-0.000)

(b) $T = \frac{1}{4}$

K	Exact (0.000)	BS (0.034)	A (0.006)	D (0.008)	FPS (0.064)	JR (0.003)
80	21.931 (0.000)	21.904 (-0.026)	21.924 (-0.007)	21.921 (-0.009)	21.959 (0.029)	21.930 (-0.001)
85	17.178 (0.000)	17.118 (-0.061)	17.172 (-0.007)	17.163 (-0.016)	17.270 (0.092)	17.185 (0.006)
90	12.659 (0.000)	12.567 (-0.092)	12.660 (0.000)	12.640 (-0.020)	12.828 (0.169)	12.669 (0.010)
95	8.604 (0.000)	8.522 (-0.082)	8.614 (0.010)	8.587 (-0.016)	8.789 (0.185)	8.604 (0.000)
100	5.277 (0.000)	5.263 (-0.014)	5.296 (0.018)	5.277 (-0.001)	5.375 (0.098)	5.266 (-0.012)
110	1.348 (0.000)	1.471 (0.123)	1.360 (0.012)	1.383 (0.035)	1.193 (-0.154)	1.348 (0.000)
115	0.545 (0.000)	0.664 (0.119)	0.545 (-0.001)	0.577 (0.031)	0.356 (-0.189)	0.552 (0.006)
125	0.056 (0.000)	0.101 (0.045)	0.043 (-0.013)	0.063 (0.006)	-0.054 (-0.110)	0.058 (0.001)
130	0.015 (0.000)	0.034 (0.020)	0.004 (-0.010)	0.015 (0.001)	-0.048 (-0.062)	0.014 (-0.001)

(c) $T = \frac{1}{2}$

K	Exact (0.000)	BS (0.051)	A (0.009)	D (0.012)	FPS (0.047)	JR (0.007)
70	33.285 (0.000)	33.265 (-0.019)	33.277 (-0.008)	33.276 (-0.009)	33.287 (0.002)	33.273 (-0.012)
80	23.948 (0.000)	23.863 (-0.085)	23.937 (-0.010)	23.926 (-0.022)	23.997 (0.050)	23.957 (0.009)
85	19.483 (0.000)	19.360 (-0.123)	19.479 (-0.004)	19.456 (-0.027)	19.576 (0.094)	19.512 (0.029)
95	11.503 (0.000)	11.391 (-0.112)	11.521 (0.018)	11.482 (-0.021)	11.634 (0.131)	11.518 (0.015)
100	8.243 (0.000)	8.203 (-0.040)	8.270 (0.027)	8.238 (-0.005)	8.340 (0.097)	8.232 (-0.011)
110	3.582 (0.000)	3.722 (0.140)	3.612 (0.031)	3.624 (0.042)	3.551 (-0.031)	3.555 (-0.027)
125	0.647 (0.000)	0.828 (0.182)	0.641 (-0.005)	0.693 (0.046)	0.511 (-0.135)	0.653 (0.006)
135	0.152 (0.000)	0.253 (0.101)	0.130 (-0.022)	0.170 (0.018)	0.041 (-0.111)	0.159 (0.007)
150	0.012 (0.000)	0.034 (0.022)	-0.004 (-0.015)	0.011 (-0.001)	-0.032 (-0.044)	0.012 (0.000)

(d) $T = 1$

K	Exact (0.000)	BS (0.076)	A (0.012)	D (0.018)	FPS (0.030)	JR (0.033)
65	40.966 (0.000)	40.932 (-0.035)	40.956 (-0.010)	40.953 (-0.013)	40.965 (-0.002)	40.896 (-0.070)
75	32.110 (0.000)	32.007 (-0.103)	32.099 (-0.011)	32.083 (-0.027)	32.131 (0.021)	32.125 (0.015)
85	23.752 (0.000)	23.585 (-0.168)	23.754 (0.002)	23.717 (-0.035)	23.815 (0.063)	23.889 (0.137)
95	16.359 (0.000)	16.216 (-0.144)	16.380 (0.021)	16.332 (-0.028)	16.444 (0.084)	16.474 (0.115)
100	13.172 (0.000)	13.084 (-0.088)	13.201 (0.029)	13.157 (-0.015)	13.251 (0.079)	13.228 (0.057)
120	4.435 (0.000)	4.657 (0.222)	4.470 (0.035)	4.500 (0.065)	4.425 (-0.011)	4.348 (-0.088)
140	1.045 (0.000)	1.306 (0.261)	1.040 (-0.004)	1.111 (0.067)	0.963 (-0.081)	1.029 (-0.016)
160	0.180 (0.000)	0.307 (0.128)	0.150 (-0.030)	0.200 (0.021)	0.102 (-0.078)	0.193 (0.014)
180	0.024 (0.000)	0.064 (0.040)	0.002 (-0.023)	0.024 (0.000)	-0.018 (-0.042)	0.031 (0.007)

Table 5.16.: Exact and approximated option prices with errors for different strikes K and maturities T in the extended Stein & Stein model for parameters as in Table 5.12 (high correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò06], **FPS** to the approximation by [FPS00], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = 2$

K	Exact (0.000)	BS (0.090)	A (0.011)	D (0.022)	FPS (0.018)	JR (0.161)
55	54.586 (0.000)	54.562 (-0.024)	54.579 (-0.007)	54.577 (-0.010)	54.582 (-0.004)	54.159 (-0.427)
65	46.463 (0.000)	46.392 (-0.071)	46.453 (-0.010)	46.443 (-0.021)	46.462 (-0.001)	46.268 (-0.195)
80	34.818 (0.000)	34.657 (-0.161)	34.816 (-0.002)	34.782 (-0.036)	34.842 (0.023)	35.339 (0.520)
94	25.105 (0.000)	24.931 (-0.174)	25.119 (0.014)	25.069 (-0.036)	25.152 (0.047)	25.727 (0.622)
115	13.758 (0.000)	13.777 (0.019)	13.791 (0.033)	13.769 (0.011)	13.803 (0.045)	13.763 (0.005)
140	5.702 (0.000)	5.995 (0.293)	5.732 (0.030)	5.784 (0.082)	5.705 (0.003)	5.415 (-0.287)
170	1.625 (0.000)	1.954 (0.329)	1.622 (-0.003)	1.709 (0.084)	1.581 (-0.044)	1.532 (-0.093)
200	0.394 (0.000)	0.590 (0.196)	0.366 (-0.028)	0.433 (0.039)	0.337 (-0.057)	0.414 (0.020)
240	0.051 (0.000)	0.113 (0.063)	0.025 (-0.025)	0.055 (0.004)	0.014 (-0.037)	0.074 (0.024)

(b) $T = 4$

K	Exact (0.000)	BS (0.104)	A (0.010)	D (0.025)	FPS (0.012)	JR (1.063)
50	65.902 (0.000)	65.877 (-0.025)	65.897 (-0.005)	65.894 (-0.008)	65.898 (-0.004)	62.896 (-3.006)
60	59.202 (0.000)	59.144 (-0.058)	59.195 (-0.007)	59.185 (-0.017)	59.199 (-0.003)	58.338 (-0.864)
80	46.390 (0.000)	46.246 (-0.144)	46.388 (-0.002)	46.357 (-0.033)	46.399 (0.009)	49.593 (3.203)
104	32.891 (0.000)	32.732 (-0.159)	32.903 (0.012)	32.857 (-0.034)	32.919 (0.027)	35.729 (2.837)
135	19.632 (0.000)	19.658 (0.026)	19.657 (0.025)	19.643 (0.011)	19.664 (0.032)	19.671 (0.040)
170	10.184 (0.000)	10.470 (0.286)	10.211 (0.027)	10.262 (0.078)	10.201 (0.017)	9.191 (-0.994)
230	2.958 (0.000)	3.360 (0.402)	2.959 (0.001)	3.062 (0.103)	2.938 (-0.020)	2.570 (-0.389)
290	0.804 (0.000)	1.071 (0.267)	0.781 (-0.023)	0.862 (0.059)	0.764 (-0.040)	0.805 (0.001)
370	0.138 (0.000)	0.244 (0.106)	0.112 (-0.026)	0.153 (0.014)	0.104 (-0.034)	0.195 (0.056)

(c) $T = 8$

K	Exact (0.000)	BS (0.115)	A (0.008)	D (0.027)	FPS (0.009)	JR (15.564)
40	81.359 (0.000)	81.350 (-0.009)	81.357 (-0.002)	81.356 (-0.003)	81.358 (-0.002)	32.613 (-48.75)
60	72.188 (0.000)	72.144 (-0.044)	72.184 (-0.004)	72.176 (-0.012)	72.186 (-0.002)	80.169 (7.981)
90	59.160 (0.000)	59.043 (-0.117)	59.159 (-0.001)	59.133 (-0.027)	59.164 (0.004)	97.023 (37.86)
126	45.471 (0.000)	45.332 (-0.139)	45.478 (0.007)	45.440 (-0.031)	45.485 (0.014)	66.600 (21.13)
180	29.716 (0.000)	29.726 (0.010)	29.733 (0.017)	29.722 (0.006)	29.738 (0.022)	29.576 (-0.140)
260	15.398 (0.000)	15.715 (0.317)	15.418 (0.020)	15.481 (0.083)	15.414 (0.016)	10.559 (-4.839)
380	5.762 (0.000)	6.242 (0.480)	5.767 (0.005)	5.885 (0.123)	5.756 (-0.006)	3.962 (-1.800)
540	1.654 (0.000)	2.013 (0.359)	1.636 (-0.018)	1.737 (0.083)	1.626 (-0.028)	1.503 (-0.151)
780	0.298 (0.000)	0.450 (0.152)	0.274 (-0.025)	0.324 (0.026)	0.269 (-0.030)	0.403 (0.104)

(d) $T = 16$

K	Exact (0.000)	BS (0.125)	A (0.006)	D (0.030)	FPS (0.007)	JR (1312.3)
40	91.306 (0.000)	91.302 (-0.004)	91.306 (-0.001)	91.305 (-0.001)	91.306 (-0.001)	-1364 (-1455)
70	84.891 (0.000)	84.867 (-0.024)	84.890 (-0.002)	84.885 (-0.007)	84.890 (-0.001)	1837 (1752)
120	74.754 (0.000)	74.681 (-0.073)	74.753 (-0.001)	74.737 (-0.017)	74.755 (0.001)	1522 (1447)
200	60.619 (0.000)	60.515 (-0.104)	60.622 (0.004)	60.595 (-0.024)	60.625 (0.007)	402.93 (342.3)
330	43.245 (0.000)	43.255 (0.010)	43.255 (0.010)	43.249 (0.004)	43.258 (0.013)	-21.236 (-64.48)
560	24.692 (0.000)	25.017 (0.325)	24.707 (0.015)	24.776 (0.084)	24.706 (0.014)	-32.520 (-57.21)
930	11.141 (0.000)	11.703 (0.562)	11.149 (0.008)	11.284 (0.143)	11.144 (0.002)	-3.181 (-14.32)
1500	3.966 (0.000)	4.473 (0.507)	3.954 (-0.011)	4.089 (0.123)	3.948 (-0.017)	2.169 (-1.797)
2600	0.813 (0.000)	1.065 (0.252)	0.789 (-0.024)	0.865 (0.051)	0.786 (-0.028)	0.963 (0.150)

Table 5.17.: Exact and approximated option prices with errors for different strikes K and maturities T in the extended Stein & Stein model for parameters as in Table 5.12 (high correlation): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò06], **FPS** to the approximation by [FPS00], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.043)	A (0.029)	D (0.002)	FPS (0.069)	JR (0.010)
70	30.558 (0.000)	30.554 (-0.004)	30.555 (-0.003)	30.556 (-0.002)	30.556 (-0.002)	30.559 (0.001)
80	20.715 (0.000)	20.675 (-0.039)	20.697 (-0.018)	20.713 (-0.001)	20.730 (0.015)	20.727 (0.012)
85	15.944 (0.000)	15.875 (-0.070)	15.924 (-0.020)	15.946 (0.002)	16.001 (0.057)	15.948 (0.004)
95	7.560 (0.000)	7.539 (-0.021)	7.595 (0.035)	7.560 (-0.000)	7.712 (0.151)	7.530 (-0.031)
100	4.479 (0.000)	4.539 (0.059)	4.542 (0.063)	4.484 (0.004)	4.607 (0.128)	4.470 (-0.009)
105	2.364 (0.000)	2.471 (0.107)	2.416 (0.052)	2.372 (0.007)	2.411 (0.047)	2.381 (0.016)
115	0.481 (0.000)	0.541 (0.060)	0.459 (-0.023)	0.479 (-0.002)	0.389 (-0.092)	0.492 (0.011)
120	0.193 (0.000)	0.219 (0.027)	0.159 (-0.034)	0.190 (-0.003)	0.100 (-0.092)	0.194 (0.001)
130	0.027 (0.000)	0.028 (0.001)	0.009 (-0.018)	0.027 (-0.000)	-0.012 (-0.039)	0.023 (-0.003)

(b) $T = \frac{1}{4}$

K	Exact (0.000)	BS (0.082)	A (0.040)	D (0.004)	FPS (0.057)	JR (0.020)
65	36.596 (0.000)	36.559 (-0.037)	36.578 (-0.018)	36.593 (-0.004)	36.585 (-0.011)	36.616 (0.020)
70	31.814 (0.000)	31.747 (-0.067)	31.788 (-0.026)	31.813 (-0.001)	31.805 (-0.009)	31.838 (0.024)
80	22.644 (0.000)	22.528 (-0.116)	22.625 (-0.019)	22.646 (0.002)	22.669 (0.026)	22.637 (-0.006)
90	14.570 (0.000)	14.508 (-0.063)	14.606 (0.036)	14.571 (0.000)	14.664 (0.094)	14.519 (-0.051)
100	8.319 (0.000)	8.400 (0.081)	8.408 (0.089)	8.325 (0.006)	8.448 (0.129)	8.287 (-0.032)
110	4.203 (0.000)	4.381 (0.178)	4.280 (0.077)	4.214 (0.012)	4.286 (0.083)	4.218 (0.016)
125	1.246 (0.000)	1.387 (0.140)	1.230 (-0.016)	1.246 (0.000)	1.204 (-0.042)	1.272 (0.026)
140	0.319 (0.000)	0.370 (0.052)	0.265 (-0.053)	0.314 (-0.005)	0.241 (-0.077)	0.322 (0.004)
160	0.047 (0.000)	0.052 (0.006)	0.018 (-0.029)	0.046 (-0.001)	0.009 (-0.038)	0.043 (-0.004)

(c) $T = \frac{1}{2}$

K	Exact (0.000)	BS (0.093)	A (0.035)	D (0.004)	FPS (0.042)	JR (0.031)
50	52.354 (0.000)	52.337 (-0.017)	52.345 (-0.009)	52.351 (-0.003)	52.346 (-0.007)	52.305 (-0.049)
60	42.954 (0.000)	42.896 (-0.058)	42.933 (-0.021)	42.952 (-0.002)	42.941 (-0.014)	42.950 (-0.004)
70	33.891 (0.000)	33.782 (-0.108)	33.868 (-0.022)	33.891 (0.000)	33.887 (-0.004)	33.970 (0.079)
85	21.750 (0.000)	21.661 (-0.089)	21.771 (0.022)	21.750 (0.000)	21.804 (0.054)	21.807 (0.057)
100	12.454 (0.000)	12.519 (0.064)	12.532 (0.078)	12.460 (0.005)	12.559 (0.105)	12.416 (-0.039)
120	5.037 (0.000)	5.248 (0.211)	5.096 (0.059)	5.050 (0.012)	5.101 (0.063)	4.999 (-0.038)
140	1.778 (0.000)	1.955 (0.177)	1.763 (-0.016)	1.781 (0.003)	1.753 (-0.026)	1.779 (0.001)
160	0.581 (0.000)	0.673 (0.092)	0.530 (-0.050)	0.577 (-0.004)	0.519 (-0.062)	0.590 (0.009)
190	0.103 (0.000)	0.125 (0.022)	0.067 (-0.036)	0.100 (-0.003)	0.061 (-0.042)	0.106 (0.003)

(d) $T = 1$

K	Exact (0.000)	BS (0.102)	A (0.026)	D (0.004)	FPS (0.031)	JR (0.310)
40	63.670 (0.000)	63.654 (-0.016)	63.663 (-0.007)	63.668 (-0.002)	63.664 (-0.006)	62.987 (-0.683)
50	54.728 (0.000)	54.678 (-0.049)	54.713 (-0.015)	54.726 (-0.002)	54.716 (-0.011)	54.557 (-0.171)
65	41.944 (0.000)	41.839 (-0.105)	41.929 (-0.014)	41.943 (-0.001)	41.940 (-0.003)	42.709 (0.765)
80	30.625 (0.000)	30.532 (-0.093)	30.638 (0.013)	30.624 (-0.001)	30.656 (0.031)	31.336 (0.711)
100	18.801 (0.000)	18.839 (0.038)	18.858 (0.057)	18.805 (0.003)	18.877 (0.075)	18.839 (0.038)
130	8.100 (0.000)	8.319 (0.219)	8.153 (0.053)	8.112 (0.012)	8.161 (0.061)	7.823 (-0.276)
160	3.229 (0.000)	3.455 (0.226)	3.227 (-0.002)	3.237 (0.008)	3.226 (-0.003)	3.108 (-0.121)
200	0.908 (0.000)	1.036 (0.128)	0.866 (-0.042)	0.907 (-0.001)	0.861 (-0.047)	0.914 (0.006)
250	0.189 (0.000)	0.231 (0.043)	0.154 (-0.035)	0.186 (-0.003)	0.151 (-0.038)	0.207 (0.019)

Table 5.18.: Exact and approximated option prices with errors for different strikes K and maturities T in the extended Stein & Stein model for parameters as in Table 5.13 (high volatility of volatility): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò06], **FPS** to the approximation by [FPS00], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = 2$

K	Exact (0.000)	BS (0.110)	A (0.020)	D (0.004)	FPS (0.023)	JR (2.977)
20	83.474 (0.000)	83.472 (-0.002)	83.473 (-0.001)	83.473 (-0.000)	83.473 (-0.001)	73.471 (-10.00)
40	67.147 (0.000)	67.108 (-0.039)	67.138 (-0.009)	67.145 (-0.002)	67.139 (-0.008)	68.322 (1.175)
55	55.567 (0.000)	55.485 (-0.082)	55.557 (-0.010)	55.565 (-0.002)	55.562 (-0.005)	63.566 (8.000)
75	41.921 (0.000)	41.836 (-0.085)	41.928 (0.007)	41.920 (-0.001)	41.938 (0.017)	47.458 (5.537)
105	26.313 (0.000)	26.350 (0.038)	26.354 (0.041)	26.316 (0.003)	26.366 (0.053)	26.345 (0.032)
150	12.517 (0.000)	12.745 (0.229)	12.560 (0.044)	12.528 (0.011)	12.568 (0.051)	11.151 (-1.366)
200	5.444 (0.000)	5.714 (0.270)	5.448 (0.005)	5.455 (0.011)	5.451 (0.007)	4.838 (-0.605)
280	1.515 (0.000)	1.687 (0.172)	1.481 (-0.034)	1.517 (0.002)	1.480 (-0.035)	1.485 (-0.029)
380	0.349 (0.000)	0.419 (0.070)	0.317 (-0.032)	0.346 (-0.002)	0.316 (-0.033)	0.393 (0.044)

(b) $T = 4$

K	Exact (0.000)	BS (0.121)	A (0.016)	D (0.004)	FPS (0.018)	JR (63.909)
20	86.372 (0.000)	86.364 (-0.008)	86.370 (-0.002)	86.371 (-0.001)	86.370 (-0.002)	-5.130 (-91.50)
30	79.702 (0.000)	79.677 (-0.025)	79.697 (-0.005)	79.701 (-0.001)	79.698 (-0.004)	279.91 (200.2)
50	67.191 (0.000)	67.128 (-0.063)	67.186 (-0.005)	67.189 (-0.001)	67.188 (-0.003)	251.78 (184.6)
70	56.222 (0.000)	56.155 (-0.068)	56.225 (0.003)	56.221 (-0.001)	56.230 (0.008)	131.65 (75.42)
110	39.187 (0.000)	39.207 (0.020)	39.212 (0.026)	39.188 (0.002)	39.219 (0.033)	32.568 (-6.619)
180	21.315 (0.000)	21.534 (0.219)	21.352 (0.037)	21.325 (0.010)	21.359 (0.044)	8.871 (-12.44)
280	9.670 (0.000)	9.989 (0.320)	9.681 (0.011)	9.683 (0.013)	9.684 (0.015)	5.787 (-3.882)
440	3.248 (0.000)	3.500 (0.253)	3.224 (-0.023)	3.254 (0.007)	3.225 (-0.023)	2.814 (-0.433)
700	0.760 (0.000)	0.877 (0.117)	0.731 (-0.030)	0.760 (-0.001)	0.730 (-0.030)	0.838 (0.078)

(c) $T = 8$

K	Exact (0.000)	BS (0.140)	A (0.012)	D (0.005)	FPS (0.013)	JR (22056)
10	95.345 (0.000)	95.343 (-0.002)	95.345 (-0.001)	95.345 (-0.000)	95.345 (-0.000)	79395 (79300)
20	90.787 (0.000)	90.775 (-0.012)	90.785 (-0.002)	90.786 (-0.001)	90.785 (-0.002)	90731 (90640)
40	82.264 (0.000)	82.229 (-0.035)	82.261 (-0.003)	82.263 (-0.001)	82.262 (-0.002)	24721 (24639)
70	71.238 (0.000)	71.195 (-0.043)	71.240 (0.001)	71.237 (-0.001)	71.242 (0.004)	1686 (1614)
130	54.616 (0.000)	54.637 (0.021)	54.631 (0.016)	54.617 (0.001)	54.636 (0.020)	-1627 (-1682)
250	34.685 (0.000)	34.891 (0.206)	34.714 (0.029)	34.694 (0.009)	34.719 (0.034)	-505.37 (-540.1)
470	17.937 (0.000)	18.306 (0.369)	17.954 (0.017)	17.952 (0.015)	17.958 (0.021)	-65.435 (-83.37)
880	7.184 (0.000)	7.545 (0.361)	7.172 (-0.012)	7.196 (0.012)	7.174 (-0.010)	-0.228 (-7.412)
1680	2.056 (0.000)	2.269 (0.213)	2.028 (-0.028)	2.059 (0.003)	2.028 (-0.027)	1.923 (-0.132)

(d) $T = 16$

K	Exact (0.000)	BS (0.164)	A (0.008)	D (0.004)	FPS (0.009)	JR ($1.4{\cdot}10^9$)
5	98.916 (0.000)	98.915 (-0.001)	98.915 (-0.000)	98.916 (-0.000)	98.915 (-0.000)	$9.6{\cdot}10^9$ ($9.6{\cdot}10^9$)
10	97.850 (0.000)	97.847 (-0.002)	97.849 (-0.000)	97.849 (-0.001)	97.849 (-0.000)	$3.0{\cdot}10^9$ ($3.0{\cdot}10^9$)
30	93.850 (0.000)	93.838 (-0.012)	93.849 (-0.001)	93.850 (-0.000)	93.849 (-0.001)	$8.0{\cdot}10^7$ ($8.0{\cdot}10^7$)
70	87.013 (0.000)	86.993 (-0.021)	87.014 (0.000)	87.012 (-0.001)	87.014 (0.001)	$-3.8{\cdot}10^7$ ($-3.8{\cdot}10^7$)
170	74.338 (0.000)	74.350 (0.012)	74.344 (0.007)	74.340 (0.002)	74.346 (0.009)	$-8.7{\cdot}10^6$ ($-8.7{\cdot}10^6$)
420	55.460 (0.000)	55.619 (0.159)	55.479 (0.019)	55.472 (0.012)	55.482 (0.022)	$-8.7{\cdot}10^5$ ($-8.7{\cdot}10^5$)
1050	33.955 (0.000)	34.341 (0.386)	33.976 (0.021)	39.974 (0.019)	33.979 (0.024)	$-4.8{\cdot}10^4$ ($-4.8{\cdot}10^4$)
2570	16.540 (0.000)	17.035 (0.495)	16.541 (0.001)	16.543 (0.003)	16.544 (0.004)	$-1.7{\cdot}10^3$ ($-1.7{\cdot}10^3$)
6350	6.023 (0.000)	6.411 (0.388)	6.001 (-0.022)	6.025 (0.002)	6.002 (-0.021)	-34.301 -40.324

Table 5.19.: Exact and approximated option prices with errors for different strikes K and maturities T in the extended Stein & Stein model for parameters as in Table 5.13 (high volatility of volatility): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **A** refers to the approximation by [Alò06], **FPS** to the approximation by [FPS00], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

5.5.4. Merton model with normal jumps

5.5.4.1. Model specification

In the model introduced by Merton [Mer76], the logarithmic discounted stock price process X is given by a Brownian motion with drift plus an independent compound Poisson process. A particularly simple and popular choice is a normal distribution of the jumps. More specifically, we have

$$X_t = \log(S_0) - \gamma t + \sigma W_t + \sum_{k=1}^{N_t} J_k, \quad t \in \mathbb{R}_+, \tag{5.41}$$

where $\sigma > 0$, W is a standard Brownian motion, $J_1, J_2, \ldots$ are independent and identically $N(\nu, \tau^2)$-distributed random variables, N is a Poisson process with intensity $\alpha \geq 0$ such that $W, J_1, J_2, \ldots, N$ are all independent. The constant $\gamma = \frac{\sigma^2}{2} + \alpha\left(e^{\nu+\frac{\tau^2}{2}} - 1\right)$.

Since $X - \log(S_0)$ is a Lévy process, the Merton model can easily be considered to be generated by a stochastic volatility model (S_0, L, V, U) in the sense of Definition 5.2.1, setting

$$\begin{aligned} L &= X - \log(S_0), \\ V &= I, \\ M &= 0, \\ U &= 0 \end{aligned} \tag{5.42}$$

with X from (5.41). V is trivially a strictly increasing, continuous process.

5.5.4.2. Check of regularity conditions

Due to the simple structure of the Merton model (5.42), all regularity conditions (cf. Table 5.1) referring to V, U, or M are automatically satisfied since $V = I$ and $U = M = 0$. Note that Condition (5.20) does not hold, but we require $c^L = \sigma^2 > 0$ in the formulation of the model.

The remaining conditions on the (exponential) moments of L_1 are given by the following

Lemma 5.5.18 (Assumptions 5.2.4 and 5.2.21(2))**.** *In the Merton model* (5.42)*, L_1 and e^{L_1} have moments of any order.*

PROOF. Relative to the truncation function $h^L(x) = 1_{[-1,1]}(x)$, the Lévy-Khintchine triplet of L is given by (b^L, c^L, F^L) with $b^L = -\gamma$, $c^L = \sigma^2$, and $F^L(\mathrm{d}x) = \alpha \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2}\left(\frac{x-\nu}{\tau}\right)^2} \mathrm{d}x$. The assertion then follows from [Sat99, Theorems 25.3 and 25.17] and the fact that the normal distribution has all moments and exponential moments. □

5.5.4.3. Moments required in the approximation

In the case of the Merton model with normal jumps, the moments used in the approximate option pricing formula from Theorem 5.4.3 are given by (cf. [Sch03, Section 5.3.8])

$$\begin{aligned}
\operatorname{Var}(L_1) &= \sigma^2 + \alpha(\nu^2 + \tau^2), \\
\operatorname{Skew}(L_1) &= \frac{\alpha(\nu^3 + 3\tau^2\nu)}{\operatorname{Var}(L_1)^{\frac{3}{2}}}, \\
\operatorname{ExKurt}(L_1) &= \frac{\alpha(\nu^4 + 6\tau^2\nu^2 + 3\tau^4)}{\operatorname{Var}(L_1)^2}, \\
\operatorname{E}(V_T) &= T, \\
\operatorname{Var}(V_T) &= 0, \\
\operatorname{Var}(M_T) &= 0, \\
\operatorname{Cov}(V_T, M_T) &= 0.
\end{aligned}$$

5.5.4.4. Numerical comparison

We assess the quality of our approximation in the Merton model with normal jumps for two different parameter sets given in Tables 5.20 and 5.21. Besides our generic benchmarks BS and JR (cf. Section 5.5.1.2), we compare our approximation to the one by [Fuk11b], indicated by F. We discuss this approach in more detail in Section 5.6.6.

The parameters from Table 5.20 are chosen such that

$$\begin{aligned}
\operatorname{E}\left(e^{L_1}\right) &= 1, \\
\operatorname{Var}(L_1) &= 0.4^2, \\
\operatorname{Skew}(L_1) &= \frac{0.1}{\sqrt{250}}, \\
\operatorname{ExKurt}(L_1) &= \frac{5}{250}, \\
\sigma &= 0.7 \cdot \sqrt{\operatorname{Var}(L_1)}.
\end{aligned}$$

The moments to which the parameters are fitted are well within the range of empirically plausible values relative to the statistical measure, cf. also Chapter 4, Section 4.4.

In contrast, the parameters from Table 5.21 are taken from [TV09b, Figure 9], where the Merton model with normal jumps is calibrated simultaneously to S&P index options with maturities $T = \frac{1}{12}, \frac{5}{12}, 1.5, 3$ years. Therefore, we consider these maturities in our experiment, for simplicity also for the parameters from Table 5.20.

In order to determine reasonable strikes for every maturity, we use the Black-Scholes volatility $\widetilde{\sigma} = \sqrt{\operatorname{Var}(L_1)}$ (cf. Section 5.5.1.1).

S_0	r	σ	α	ν	τ
100	0	0.280	39.0	−0.00165	0.0457

Table 5.20.: Merton model parameters (statistical measure)

S_0	r	σ	α	ν	τ
100	0	0.09	0.39	−0.12	0.15

Table 5.21.: Merton model parameters (risk-neutral measure)

We use the representation of the characteristic function of $X_T = \log(S_T)$ from [HKK06, Section 5.2] to compute the exact option price by the Laplace transform method (cf. Section 5.5.1.3).

Table 5.22 shows the exact price, the different approximations, and their errors for parameters from Table 5.20, Table 5.23 for parameters from Table 5.21.

For parameters from Table 5.20, our approximation is precise to the quoted number of digits for the last three maturities, and the MAEOS for the remaining maturity $T = \frac{1}{12}$ amounts to 0.001. In terms of the MAEOS, we clearly outperform the alternative approximations. Note that the approximation F is outperformed by the mere Black-Scholes price BS for the first two maturities, and its MAEOS is roughly at the same level as the one of BS for the remaining two maturities. For all but the first maturity, JR gets outperformed by BS as well.

The picture its different for the experiment with parameters from Table 5.21. For maturity $T = \frac{1}{12}$, ours and the approximations F and JR are outperformed by BS in terms of the MAEOS. Our approximation shows a clearly lower MAEOS than the other approximations for maturities $T = 1.5$ and $T = 3$. The MAEOS of our approximation decreases from a level of 0.623 for maturity $T = \frac{1}{12}$ to 0.035 for maturity $T = 3$. In the first case, our approximation – but also the ones of F and JR – even lead to negative prices for some strikes, which may of course happen for first- or second-order approximations. The fact that our approximation – as a perturbation around the Black-Scholes model – works better for the longer maturities is not surprising because – by its independent increments and the central limit theorem – the marginal law of a Lévy process tends to a normal distribution as the point in time increases. In terms of $\operatorname{Skew}(L_1)$ and $\operatorname{ExKurt}(L_1)$, the parameters from Table 5.21 lead to a distribution of L_1 that is much more distant from the normal distribution than for the first parameter set. Indeed, we have $\operatorname{Skew}(L_1) = \frac{-17.96}{\sqrt{250}}$ and $\operatorname{ExKurt}(L_1) = \frac{707.40}{250}$ for the second parameter set in contrast to $\operatorname{Skew}(L_1) = \frac{0.1}{\sqrt{250}}$ and $\operatorname{ExKurt}(L_1) = \frac{5}{250}$ for the first one. The quite extreme values of skewness and excess kurtosis for the risk-neutral parameters probably result from the fact that – since the law of L_1 determines already the law of L – geometric Lévy models are very difficult to calibrate to short- and long-termed options at once, which results in extreme parameter choices.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.018)	D (0.001)	F (0.019)	JR (0.014)
75	25.030 (0.000)	25.021 (-0.009)	25.030 (0.000)	25.022 (-0.008)	25.045 (0.015)
80	20.122 (0.000)	20.105 (-0.018)	20.123 (0.001)	20.109 (-0.014)	20.133 (0.010)
85	15.401 (0.000)	15.383 (-0.017)	15.401 (0.000)	15.390 (-0.010)	15.386 (-0.015)
90	11.072 (0.000)	11.074 (0.002)	11.071 (-0.001)	11.082 (0.010)	11.034 (-0.038)
100	4.556 (0.000)	4.603 (0.047)	4.556 (-0.000)	4.601 (0.044)	4.552 (-0.004)
105	2.594 (0.000)	2.633 (0.039)	2.594 (-0.001)	2.624 (0.029)	2.614 (0.020)
115	0.685 (0.000)	0.679 (-0.007)	0.685 (-0.000)	0.667 (-0.018)	0.703 (0.018)
125	0.149 (0.000)	0.131 (-0.018)	0.150 (0.001)	0.125 (-0.024)	0.148 (-0.001)
135	0.029 (0.000)	0.020 (-0.009)	0.029 (0.000)	0.018 (-0.011)	0.025 (-0.004)

(b) $T = \frac{5}{12}$

K	Exact (0.000)	BS (0.008)	D (0.000)	F (0.009)	JR (0.029)
50	50.023 (0.000)	50.020 (-0.003)	50.023 (0.000)	50.021 (-0.002)	50.065 (0.043)
60	40.186 (0.000)	40.179 (-0.007)	40.186 (0.000)	40.181 (-0.004)	40.207 (0.021)
70	30.827 (0.000)	30.820 (-0.007)	30.827 (0.000)	30.825 (-0.002)	30.786 (-0.041)
80	22.460 (0.000)	22.462 (0.002)	22.460 (-0.000)	22.467 (0.007)	22.388 (-0.071)
95	12.688 (0.000)	12.710 (0.021)	12.688 (-0.000)	12.707 (0.018)	12.661 (-0.028)
115	5.089 (0.000)	5.109 (0.020)	5.088 (-0.000)	5.094 (0.006)	5.118 (0.029)
135	1.804 (0.000)	1.807 (0.002)	1.804 (-0.000)	1.791 (-0.013)	1.826 (0.022)
160	0.447 (0.000)	0.440 (-0.008)	0.448 (0.000)	0.431 (-0.016)	0.449 (0.002)
190	0.079 (0.000)	0.074 (-0.005)	0.079 (0.000)	0.071 (-0.008)	0.076 (-0.003)

(c) $T = 1.5$

K	Exact (0.000)	BS (0.006)	D (0.000)	F (0.005)	JR (0.160)
25	75.017 (0.000)	75.016 (-0.001)	75.017 (0.000)	75.017 (-0.000)	75.520 (0.503)
35	65.165 (0.000)	65.162 (-0.003)	65.165 (0.000)	65.164 (-0.001)	65.231 (0.066)
50	51.206 (0.000)	51.203 (-0.003)	51.206 (0.000)	51.205 (-0.000)	50.819 (-0.387)
65	39.041 (0.000)	39.043 (0.003)	39.041 (-0.000)	39.045 (0.004)	38.718 (-0.323)
90	23.780 (0.000)	23.795 (0.015)	23.780 (0.000)	23.787 (0.007)	23.735 (-0.045)
120	12.687 (0.000)	12.706 (0.019)	12.687 (-0.000)	12.688 (0.001)	12.750 (0.063)
160	5.458 (0.000)	5.469 (0.011)	5.458 (-0.000)	5.447 (-0.011)	5.505 (0.047)
230	1.328 (0.000)	1.327 (-0.001)	1.328 (0.000)	1.313 (-0.015)	1.335 (0.008)
310	0.302 (0.000)	0.299 (-0.003)	0.302 (0.000)	0.293 (-0.009)	0.301 (-0.002)

(d) $T = 3$

K	Exact (0.000)	BS (0.006)	D (0.000)	F (0.005)	JR (2.177)
10	90.003 (0.000)	90.002 (-0.000)	90.003 (-0.000)	90.003 (0.000)	100.55 (10.54)
20	80.102 (0.000)	80.101 (-0.001)	80.102 (0.000)	80.101 (-0.000)	78.513 (-1.589)
30	70.606 (0.000)	70.604 (-0.002)	70.606 (0.000)	70.605 (-0.000)	65.750 (-4.856)
50	53.922 (0.000)	53.924 (0.002)	53.922 (-0.000)	53.924 (0.002)	51.776 (-2.147)
80	35.570 (0.000)	35.582 (0.013)	35.570 (0.000)	35.573 (0.003)	35.604 (0.034)
120	20.765 (0.000)	20.785 (0.020)	20.765 (0.000)	20.763 (-0.002)	21.057 (0.292)
190	8.785 (0.000)	8.800 (0.015)	8.785 (-0.000)	8.772 (-0.013)	8.896 (0.112)
300	2.754 (0.000)	2.758 (0.004)	2.754 (0.000)	2.738 (-0.016)	2.774 (0.020)
470	0.635 (0.000)	0.634 (-0.001)	0.635 (0.000)	0.625 (-0.010)	0.636 (0.000)

Table 5.22.: Exact and approximated option prices with errors for different strikes K and maturities T in the Merton model with normal jumps for parameters as in Table 5.20 (statistical measure): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **F** refers to the approximation by [Fuk11b], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.229)	D (0.623)	F (0.425)	JR (0.476)
92	8.215 (0.000)	8.043 (-0.172)	8.977 (0.762)	8.359 (0.145)	8.878 (0.663)
94	6.259 (0.000)	6.144 (-0.114)	6.834 (0.576)	6.684 (0.425)	6.921 (0.662)
96	4.342 (0.000)	4.393 (0.052)	4.428 (0.086)	5.018 (0.676)	4.479 (0.137)
98	2.590 (0.000)	2.893 (0.302)	2.148 (-0.442)	3.297 (0.707)	2.011 (-0.579)
100	1.242 (0.000)	1.727 (0.485)	0.332 (-0.910)	1.653 (0.411)	0.333 (-0.909)
102	0.460 (0.000)	0.924 (0.464)	-0.729 (-1.189)	0.385 (-0.075)	-0.211 (-0.671)
104	0.141 (0.000)	0.439 (0.297)	-0.846 (-0.987)	-0.306 (-0.447)	-0.033 (-0.174)
106	0.050 (0.000)	0.184 (0.134)	-0.276 (-0.326)	-0.481 (-0.532)	0.244 (0.193)
108	0.029 (0.000)	0.068 (0.039)	0.361 (0.332)	-0.379 (-0.408)	0.322 (0.293)

(b) $T = \frac{5}{12}$

K	Exact (0.000)	BS (0.323)	D (0.163)	F (0.286)	JR (0.066)
75	25.150 (0.000)	25.004 (-0.146)	25.160 (0.010)	25.037 (-0.113)	25.148 (-0.002)
80	20.299 (0.000)	20.031 (-0.268)	20.429 (0.130)	20.185 (-0.114)	20.430 (0.131)
90	10.902 (0.000)	10.644 (-0.258)	11.014 (0.111)	11.204 (0.302)	10.881 (-0.021)
95	6.596 (0.000)	6.780 (0.183)	6.657 (0.061)	7.149 (0.552)	6.385 (-0.211)
100	3.183 (0.000)	3.860 (0.678)	3.118 (-0.065)	3.696 (0.514)	3.057 (-0.126)
105	1.173 (0.000)	1.949 (0.775)	0.831 (-0.342)	1.286 (0.112)	1.197 (0.024)
110	0.356 (0.000)	0.870 (0.514)	-0.095 (-0.451)	0.044 (-0.312)	0.412 (0.056)
120	0.050 (0.000)	0.122 (0.073)	0.124 (0.075)	-0.305 (-0.354)	0.065 (0.015)
125	0.025 (0.000)	0.039 (0.014)	0.243 (0.218)	-0.179 (-0.204)	0.031 (0.005)

(c) $T = 1.5$

K	Exact (0.000)	BS (0.365)	D (0.055)	F (0.184)	JR (0.080)
60	40.112 (0.000)	40.011 (-0.101)	40.125 (0.012)	40.057 (-0.055)	40.144 (0.031)
70	30.429 (0.000)	30.152 (-0.277)	30.478 (0.049)	30.391 (-0.038)	30.550 (0.121)
80	21.270 (0.000)	20.891 (-0.379)	21.269 (-0.001)	21.345 (0.075)	21.245 (-0.025)
90	13.091 (0.000)	13.060 (-0.030)	13.087 (-0.003)	13.331 (0.241)	12.857 (-0.234)
100	6.660 (0.000)	7.317 (0.657)	6.723 (0.062)	7.006 (0.346)	6.568 (-0.092)
110	2.696 (0.000)	3.691 (0.995)	2.666 (-0.030)	2.869 (0.173)	2.814 (0.118)
125	0.495 (0.000)	1.114 (0.619)	0.298 (-0.197)	0.215 (-0.280)	0.568 (0.073)
140	0.085 (0.000)	0.285 (0.200)	0.048 (-0.038)	-0.240 (-0.325)	0.072 (-0.013)
160	0.011 (0.000)	0.039 (0.028)	0.115 (0.104)	-0.115 (-0.125)	-0.002 (-0.012)

(d) $T = 3$

K	Exact (0.000)	BS (0.423)	D (0.035)	F (0.129)	JR (0.155)
50	50.108 (0.000)	50.021 (-0.086)	50.116 (0.009)	50.073 (-0.035)	49.912 (-0.196)
60	40.407 (0.000)	40.186 (-0.221)	40.430 (0.024)	40.384 (-0.023)	40.514 (0.108)
70	31.164 (0.000)	30.840 (-0.324)	31.170 (0.006)	31.200 (0.036)	31.574 (0.409)
80	22.722 (0.000)	22.501 (-0.221)	22.703 (-0.019)	22.827 (0.105)	22.977 (0.255)
100	9.683 (0.000)	10.334 (0.651)	9.716 (0.033)	9.896 (0.213)	9.432 (-0.251)
115	4.079 (0.000)	5.167 (1.088)	4.089 (0.009)	4.200 (0.121)	3.962 (-0.118)
135	0.998 (0.000)	1.844 (0.845)	0.891 (-0.108)	0.809 (-0.189)	1.034 (0.036)
160	0.143 (0.000)	0.455 (0.312)	0.099 (-0.044)	-0.153 (-0.296)	0.163 (0.020)
190	0.015 (0.000)	0.078 (0.063)	0.081 (0.066)	-0.131 (-0.146)	0.017 (0.002)

Table 5.23.: Exact and approximated option prices with errors for different strikes K and maturities T in the Merton model with normal jumps for parameters as in Table 5.21 (risk-neutral measure): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **F** refers to the approximation by [Fuk11b], and **JR** to the approximation by [JR82]. **D** refers to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

5.5.5. NIG-CIR model

5.5.5.1. Model specification

In the NIG-CIR model from [CGMY03], the logarithmic discounted stock price process $X = \log(S)$ is given by

$$\begin{aligned} X_t &= \log(S_0) + L_{V_t} + \rho\theta \int_0^t \sqrt{y_s}\,\mathrm{d}W_s - \frac{1}{2}\rho^2\theta^2 V_t, \\ V_t &= \int_0^t y_s\,\mathrm{d}s, \\ dy_t &= \kappa(\eta - y_t)\,\mathrm{d}t + \theta\sqrt{y_t}\,\mathrm{d}W_t, \quad y_0 > 0, \end{aligned} \tag{5.43}$$

$t \in \mathbb{R}_+$, $\rho \in \mathbb{R}$, κ, η, $\theta > 0$, W is a standard Brownian motion, and X is an NIG Lévy process with parameters $\alpha > 0$, $\beta \in (-\alpha, \alpha)$, $\delta > 0$, and $\mu = \delta\left(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}\right)$. I.e., the cumulant generating function of L is given by

$$\kappa(z) = \mu z + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}\right), \quad z \in \{y \in \mathbb{C} : |\beta + \mathrm{Re}\,(y)| < \alpha\}.$$

We can directly consider the NIG-CIR model to be generated by a stochastic volatility model (S_0, L, V, U) from Definition 5.2.1 with L and V as in (5.43), and setting

$$U_t = \rho\theta \int_0^t \sqrt{y_s}\,\mathrm{d}W_s - \frac{1}{2}\rho^2\theta^2 V_t, \quad t \in \mathbb{R}_+, \tag{5.44}$$

and hence

$$M_t = \rho\theta \int_0^t \sqrt{y}_s\,\mathrm{d}W_s, \quad t \in \mathbb{R}_+. \tag{5.45}$$

Since the process V for the NIG-CIR model and the Heston model from Section 5.5.2.1 coincide, Lemma 5.5.3 yields that V is almost surely strictly increasing, as it is required in the Definition 5.2.1 of a stochastic volatility model in our sense.

5.5.5.2. Check of regularity conditions

It is obvious that Assumptions 5.2.9, 5.2.16, and 5.2.26 hold. The condition on the decay of the Laplace transform of V_T from Assumption 5.2.21(1a) is given by Lemma 5.5.6 since the process V is the same in the Heston model. By the same reason, a criterion for Assumption 5.2.21(3) is provided by Lemma 5.5.7. We treat the remaining assumptions in a series of lemmas.

Lemma 5.5.19 (Assumption 5.2.4)**.** *In the NIG-CIR model from* (5.43)*, L_1 has moments of any order.*

PROOF. By Lemma 5.5.23 below, there is $\varepsilon > 0$ such that $\mathrm{E}\left(e^{\pm\varepsilon L_1}\right) < \infty$. This implies the existence of all moments of L_1. □

Lemma 5.5.20 (Assumption 5.2.8)**.** *M from* (5.45) *is a martingale.*

PROOF. For all $t \in \mathbb{R}_+$, we have $\mathrm{E}\left(\rho^2\theta^2 \int_0^t y_s\,\mathrm{d}s\right) = \rho^2\theta^2\mathrm{E}\left(V_t\right) < \infty$ by Lemma 5.5.25 below, which implies that the local martingale M is a martingale. □

Lemma 5.5.21 (Assumption 5.2.11)**.** *For U from* (5.44)*, the process e^{U^λ} defined in* (5.12) *is a martingale for all $\lambda \in [0,1]$.*

PROOF. This follows as in the proof of Lemma 5.5.5 with ρ replaced by $\rho\theta$. □

Lemma 5.5.22 (Assumption 5.2.211a)**.** *For L from the NIG-CIR model* (5.43)*, Condition* (5.20) *holds.*

PROOF. Cf. Section 5.7.6. □

Lemma 5.5.23 (Criterion for Assumption 5.2.21(2))**.** *For L from the NIG-CIR model* (5.43)*, we have for all $a \in (-\beta-\alpha, -\beta+\alpha)$ that*

$$\mathrm{E}\left(e^{aL_1}\right) < \infty.$$

PROOF. This follows from the tail behavior of the density of the NIG distribution stated in [Sch03, Section 5.3.8]. □

Lemma 5.5.24 (Criterion for Assumption 5.2.21(4))**.** *Consider U and M from* (5.44) *and* (5.45)*. For all $t \in \mathbb{R}_+$ and all $a \in \mathbb{R}$ with $a\rho < \frac{\kappa}{2\theta^2}$, we have*

$$\mathrm{E}\left(e^{aM_t}\right) < \infty.$$

Moreover, for all $t \in \mathbb{R}_+$ and all $b \in \mathbb{R}$ with $b\rho < \frac{\kappa}{4\theta^2}$ and $-b\rho^2 \leq \frac{\kappa^2}{2\theta^4}$, we have

$$\mathrm{E}\left(e^{bU_t}\right) < \infty.$$

PROOF. Cf. Lemma 5.5.8 with ρ replaced by $\rho\theta$. □

Lemma 5.5.25 (Assumptions 5.2.15 and 5.2.25)**.** *For V from* (5.43) *and M from* (5.45)*, for all $t \in \mathbb{R}_+$, the random variables M_t, $\langle M,M\rangle_t$, and V_t have moments of any order.*

PROOF. This follows as in the proof of Lemma 5.5.9, using Lemmas 5.5.7 and 5.5.24. □

S_0	y_0	r	α	β	δ	κ	η	θ	ρ
100	0.0485	0.0456	90.1	−16.0	85.9	2.54	0.0485	0.5386	0

Table 5.24.: NIG-CIR model parameters (statistical measure)

5.5.5.3. Moments required in the approximation

For the NIG-CIR model from (5.43), the moments required in our approximation to option prices from Theorem 5.4.3 are given by

$$\begin{aligned}
\operatorname{Var}(L_1) &= \frac{\alpha^2\delta}{(\alpha^2-\beta^2)^{\frac{3}{2}}},\\
\operatorname{Skew}(L_1) &= \frac{3\beta}{\alpha\sqrt{\delta}\,(\alpha^2-\beta^2)^{\frac{1}{4}}},\\
\operatorname{ExKurt}(L_1) &= \frac{3\left(\alpha^2+4\beta^2\right)}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}},\\
\mathrm{E}(V_T) &= \eta T + \frac{(y_0-\eta)(1-e^{-\kappa T})}{\kappa},\\
\operatorname{Var}(V_T) &= \frac{\theta^2}{2\kappa^3}\Big(2y_0-5\eta-2y_0e^{-2\kappa T}-4y_0e^{-\kappa T}\kappa T\\
&\qquad +\eta e^{-2\kappa T}+4\eta e^{-\kappa T}(\kappa T+1)+2\eta T\kappa\Big),\\
\operatorname{Var}(M_T) &= \rho^2\theta^2\mathrm{E}(V_T),\\
\operatorname{Cov}(V_T,M_T) &= \rho\left(\int_0^T \operatorname{Cov}(y_T,y_t)\,\mathrm{d}t+\kappa\operatorname{Var}(V_T)\right),
\end{aligned}$$

where

$$\int_0^T \operatorname{Cov}(y_T,y_t)\,\mathrm{d}t = \frac{\theta^2}{2\kappa^2}\left(2e^{-\kappa T}(y_0\kappa T-y_0-\eta\kappa T)+e^{-2\kappa T}(2y_0-\eta)+\eta\right).$$

The moments of L_1 are to be found in [Sch03, Section 5.3.8]. The moment structure of (V_T,M_T) follows from the corresponding formulas in the Heston model, cf. Section 5.5.2.3, since the process V coincides in the Heston and NIG-CIR model, and the process M differs only by the constant factor θ.

5.5.5.4. Numerical comparison

We assess the quality of our approximation in the NIG-CIR model for two different parameter sets given in Tables 5.24 and 5.25. The first parameter set is taken from [KMK11, Section 3.3],

S_0	y_0	r	α	β	δ	κ	η	θ	ρ
100	1	0	18.4815	-4.8412	0.4685	0.5391	1.5746	1.8772	0

Table 5.25.: NIG-CIR model parameters (risk-neutral measure)

where the NIG-CIR model is fitted to a time series of daily DAX returns, i.e., this parameter choice corresponds to dynamics of the NIG-CIR model relative to the statistical measure, where we ensured the martingale property of the price process by proper drift correction. The second parameter set is taken from [Sch03, Table 7.3], where the NIG-CIR model is calibrated to call options on the S&P 500 index with different strikes and maturities. Hence, the parameters of Table 5.25 correspond to dynamics of the NIG-CIR model relative to an implied risk-neutral measure. We compare our approximation only to the generic benchmarks BS and JR since no more specific approximations are available for the NIG-CIR model in the literature.

In order to determine reasonable strikes for every maturity, we use the Black-Scholes volatility $\widetilde{\sigma} = \sqrt{\eta \operatorname{Var}(L_1)}$, cf. Section 5.5.1.1. Recall from Section 5.5.2.4 that $\sqrt{\eta}$ corresponds to the long-term mean volatility.

We use the representation of the characteristic function of $X_T = \log(S_T)$ from [CGMY03, Section 4] to compute the exact option prices by the Laplace transform method (cf. Section 5.5.1.3).

As we see from Section 5.5.5.2, the only regularity conditions whose validity depends on the model parameters are Assumptions 5.2.21(2), 5.2.21(3), and 5.2.21(4). For both parameters sets, by Lemma 5.5.23, the choices of α and β are such that we can choose the parameter R in the integral representation of the call option payoff function from (5.34) such that Assumption 5.2.21(2) holds. Moreover, Assumption 5.2.21(4) is satisfied by Lemma 5.5.24 since we have $\rho = 0$ for both parameter sets. Finally, since $\kappa^{\lambda}(1) = 0$ for all $\lambda \in [0,1]$ by construction, the continuity of $(\lambda, r) \mapsto \kappa^{\lambda}(r)$ (cf. the proof of Lemma 5.7.3) allows to choose $R > 1$ such that also Assumption 5.2.21(3) is satisfied because a small positive exponential moment of V_T always exists by Lemma 5.5.7.

Tables 5.26 and 5.27 show the exact price, the different approximations, and their errors for parameters from Table 5.24, and Tables 5.28 and 5.29 for parameters from Table 5.25.

For the parameters from Table 5.24, our approximation outperforms BS and JR in terms of the MAEOS for all maturities under consideration. The MAEOS is 0.002 for $T = \frac{1}{12}$, increases to 0.040 up to $T = 1$, and then decreases to a level of 0.011 up to $T = 8$. For $T = 16$, it jumps to 0.781; we have no explanation for this big difference, but it might occur due to numerical errors in the evaluation of the exact option price. For this first parameter set, we have $\operatorname{Skew}(L_1) = -\frac{0.0965}{\sqrt{250}}$ and $\operatorname{ExKurt}(L_1) = \frac{0.111}{250}$. In terms of these moments, the Lévy process L can hence be considered as very close to Brownian motion for this parameter choice.

The picture is different for the parameters from Table 5.25, for which we have $\operatorname{Skew}(L_1) = -\frac{4.30}{\sqrt{250}}$ and $\operatorname{ExKurt}(L_1) = \frac{114}{250}$. This explains that for the second parameter set, the MAEOS are higher than for the first set. E.g., for $T = \frac{1}{12}$, it amounts to 0.002 in the first and to 0.071 in the second case. For $T = \frac{1}{4}$, it goes down to 0.037, then rises to a level of 0.044 for $T = \frac{1}{2}$, and increases to 0.654 up to maturity $T = 16$. For all maturities but the first one, our approximation clearly outperforms the alternative ones.

Note that intuitively, the error of our approximation is driven by two factors: for early maturities, the marginal law of the Lévy process L (which is, of course, evaluated at a random time in the NIG-CIR model) is still away from the normal distribution and gets closer to it for later points in time. The stochastic volatility process is almost constant for short maturities, its stochasticity takes effect afterwards, and it reaches its stationary distribution for later points in time. In particular for the second parameter set, there seems to be an interplay of these two effects. Moreover, note that the parameter $\theta = 0.5386$ for the first and $\theta = 1.8772$ for the second parameter choice. Hence, in the second case not only L is more distant from Brownian motion, but also the volatility process is "more stochastic".

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.024)	D (0.002)	JR (0.008)
84	16.332 (0.000)	16.323 (-0.009)	16.332 (-0.000)	16.337 (0.004)
88	12.401 (0.000)	12.376 (-0.026)	12.404 (0.003)	12.409 (0.008)
92	8.621 (0.000)	8.587 (-0.034)	8.623 (0.002)	8.615 (-0.006)
96	5.248 (0.000)	5.250 (0.002)	5.246 (-0.002)	5.226 (-0.022)
100	2.669 (0.000)	2.726 (0.057)	2.669 (-0.000)	2.666 (-0.003)
104	1.116 (0.000)	1.170 (0.054)	1.113 (-0.002)	1.130 (0.014)
108	0.395 (0.000)	0.409 (0.014)	0.392 (-0.003)	0.405 (0.009)
112	0.125 (0.000)	0.116 (-0.008)	0.125 (0.001)	0.126 (0.001)
116	0.036 (0.000)	0.027 (-0.009)	0.039 (0.002)	0.034 (-0.002)

(b) $T = \frac{1}{4}$

K	Exact (0.000)	BS (0.060)	D (0.009)	JR (0.066)
75	25.886 (0.000)	25.860 (-0.026)	25.887 (0.001)	25.933 (0.047)
80	21.019 (0.000)	20.965 (-0.054)	21.031 (0.012)	21.081 (0.061)
90	11.822 (0.000)	11.790 (-0.032)	11.821 (-0.002)	11.703 (-0.119)
95	7.893 (0.000)	7.967 (0.075)	7.879 (-0.013)	7.746 (-0.146)
100	4.786 (0.000)	4.957 (0.171)	4.777 (-0.009)	4.734 (-0.052)
110	1.400 (0.000)	1.474 (0.074)	1.386 (-0.014)	1.486 (0.086)
115	0.716 (0.000)	0.706 (-0.010)	0.712 (-0.004)	0.784 (0.068)
125	0.184 (0.000)	0.127 (-0.056)	0.198 (0.015)	0.191 (0.007)
130	0.094 (0.000)	0.049 (-0.045)	0.105 (0.011)	0.086 (-0.007)

(c) $T = \frac{1}{2}$

K	Exact (0.000)	BS (0.116)	D (0.025)	JR (0.177)
65	36.503 (0.000)	36.472 (-0.032)	36.502 (-0.002)	36.664 (0.160)
75	26.900 (0.000)	26.806 (-0.094)	26.930 (0.030)	27.097 (0.197)
80	22.245 (0.000)	22.139 (-0.106)	22.279 (0.033)	22.219 (-0.026)
90	13.672 (0.000)	13.712 (0.040)	13.654 (-0.019)	13.166 (-0.506)
100	7.040 (0.000)	7.330 (0.290)	7.004 (-0.036)	6.794 (-0.246)
110	3.129 (0.000)	3.364 (0.235)	3.093 (-0.036)	3.312 (0.183)
125	0.867 (0.000)	0.805 (-0.062)	0.881 (0.014)	1.055 (0.188)
135	0.380 (0.000)	0.268 (-0.112)	0.420 (0.040)	0.443 (0.063)
150	0.119 (0.000)	0.043 (-0.076)	0.137 (0.018)	0.097 (-0.021)

(d) $T = 1$

K	Exact (0.000)	BS (0.152)	D (0.040)	JR (0.649)
55	47.500 (0.000)	47.460 (-0.040)	47.497 (-0.002)	48.234 (0.734)
65	38.096 (0.000)	37.991 (-0.104)	38.130 (0.034)	38.953 (0.858)
75	29.000 (0.000)	28.871 (-0.128)	29.048 (0.049)	28.630 (-0.370)
90	16.852 (0.000)	16.993 (0.141)	16.812 (-0.040)	15.230 (-1.622)
100	10.606 (0.000)	10.985 (0.380)	10.536 (-0.069)	9.708 (-0.897)
120	3.588 (0.000)	3.833 (0.245)	3.535 (-0.053)	4.186 (0.598)
135	1.570 (0.000)	1.530 (-0.040)	1.587 (0.016)	2.133 (0.563)
155	0.566 (0.000)	0.398 (-0.168)	0.635 (0.069)	0.732 (0.167)
180	0.183 (0.000)	0.065 (-0.118)	0.211 (0.029)	0.150 (-0.032)

Table 5.26.: Exact and approximated option prices with errors for different strikes K and maturities T in the NIG-CIR model for parameters as in Table 5.24 (statistical measure): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **JR** refers to the approximation by [JR82], and **D** to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = 2$

K	Exact (0.000)	BS (0.164)	D (0.037)	JR (2.141)
45	58.976 (0.000)	58.935 (-0.041)	58.975 (-0.001)	62.789 (3.813)
55	49.995 (0.000)	49.897 (-0.098)	50.021 (0.026)	53.075 (3.080)
70	37.055 (0.000)	36.934 (-0.121)	37.094 (0.040)	34.244 (-2.810)
85	25.515 (0.000)	25.601 (0.086)	25.494 (-0.021)	20.493 (-5.022)
105	13.872 (0.000)	14.307 (0.436)	13.802 (-0.070)	12.237 (-1.635)
130	5.845 (0.000)	6.169 (0.324)	5.790 (-0.055)	7.195 (1.350)
155	2.470 (0.000)	2.453 (-0.017)	2.483 (0.014)	3.684 (1.215)
190	0.827 (0.000)	0.629 (-0.198)	0.898 (0.071)	1.147 (0.320)
230	0.283 (0.000)	0.129 (-0.154)	0.319 (0.036)	0.255 (-0.028)

(b) $T = 4$

K	Exact (0.000)	BS (0.155)	D (0.022)	JR (8.556)
35	70.886 (0.000)	70.854 (-0.032)	70.886 (-0.000)	94.567 (23.68)
45	62.698 (0.000)	62.623 (-0.075)	62.711 (0.013)	72.453 (9.755)
60	50.868 (0.000)	50.761 (-0.107)	50.892 (0.024)	36.119 (-14.75)
80	36.729 (0.000)	36.765 (0.036)	36.724 (-0.005)	18.028 (-18.70)
110	20.817 (0.000)	21.197 (0.380)	20.777 (-0.040)	17.342 (-3.475)
145	10.178 (0.000)	10.550 (0.372)	10.140 (-0.038)	13.648 (3.470)
190	4.149 (0.000)	4.187 (0.037)	4.151 (0.001)	6.720 (2.571)
255	1.315 (0.000)	1.123 (-0.192)	1.362 (0.047)	1.888 (0.573)
340	0.381 (0.000)	0.220 (-0.161)	0.407 (0.026)	0.350 (-0.032)

(c) $T = 8$

K	Exact (0.000)	BS (0.136)	D (0.011)	JR (92.453)
20	86.126 (0.000)	86.118 (-0.008)	86.125 (-0.001)	479.45 (393.3)
35	75.862 (0.000)	75.816 (-0.046)	75.866 (0.005)	43.283 (-32.58)
55	62.865 (0.000)	62.793 (-0.072)	62.875 (0.009)	-150.89 (-213.8)
70	53.993 (0.000)	53.966 (-0.027)	53.997 (0.004)	-108.06 (-162.1)
120	31.393 (0.000)	31.679 (0.286)	31.376 (-0.017)	27.763 (-3.630)
180	16.346 (0.000)	16.702 (0.356)	16.327 (-0.019)	33.748 (17.40)
260	7.351 (0.000)	7.461 (0.110)	7.346 (-0.005)	15.368 (8.017)
400	2.274 (0.000)	2.119 (-0.155)	2.297 (0.023)	3.567 (1.293)
590	0.657 (0.000)	0.494 (-0.163)	0.676 (0.019)	0.683 (0.026)

(d) $T = 16$

K	Exact (0.000)	BS (0.861)	D (0.781)	JR (4876.0)
15	89.603 (0.000)	92.777 (3.175)	92.783 (3.180)	6817 (6727)
25	86.371 (0.000)	88.024 (1.653)	88.044 (1.674)	-19234 (-19320)
45	79.912 (0.000)	78.947 (-0.965)	78.990 (-0.922)	-14028 (-14108)
80	65.697 (0.000)	65.089 (-0.608)	65.084 (-0.613)	-2696 (-2762)
140	46.871 (0.000)	47.339 (0.467)	47.150 (0.279)	575.62 (528.7)
250	27.532 (0.000)	28.059 (0.527)	27.725 (0.193)	383.73 (356.2)
440	13.136 (0.000)	13.230 (0.093)	13.045 (-0.091)	86.554 (73.42)
770	4.891 (0.000)	4.760 (-0.131)	4.841 (-0.050)	13.154 (8.263)
1360	1.360 (0.000)	1.228 (-0.132)	1.392 (0.033)	1.815 (0.455)

Table 5.27.: Exact and approximated option prices with errors for different strikes K and maturities T in the NIG-CIR model for parameters as in Table 5.24 (statistical measure): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **JR** refers to the approximation by [JR82], and **D** to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = \frac{1}{12}$

K	Exact (0.000)	BS (0.079)	D (0.071)	JR (0.046)
84	16.030 (0.000)	16.000 (-0.030)	16.013 (-0.017)	16.013 (-0.017)
88	12.078 (0.000)	12.006 (-0.072)	12.114 (0.036)	12.113 (0.035)
92	8.205 (0.000)	8.084 (-0.122)	8.345 (0.139)	8.323 (0.117)
96	4.565 (0.000)	4.538 (-0.027)	4.572 (0.007)	4.507 (-0.059)
100	1.681 (0.000)	1.946 (0.265)	1.534 (-0.146)	1.577 (-0.104)
104	0.435 (0.000)	0.594 (0.159)	0.291 (-0.144)	0.419 (-0.015)
108	0.121 (0.000)	0.124 (0.003)	0.160 (0.038)	0.166 (0.045)
112	0.039 (0.000)	0.018 (-0.021)	0.119 (0.080)	0.063 (0.024)
116	0.013 (0.000)	0.002 (-0.012)	0.042 (0.029)	0.014 (0.000)

(b) $T = \frac{1}{4}$

K	Exact (0.000)	BS (0.092)	D (0.037)	JR (0.040)
80	20.082 (0.000)	20.011 (-0.071)	20.094 (0.012)	20.121 (0.039)
85	15.221 (0.000)	15.087 (-0.134)	15.286 (0.065)	15.291 (0.070)
90	10.562 (0.000)	10.426 (-0.137)	10.615 (0.052)	10.527 (-0.036)
95	6.359 (0.000)	6.405 (0.046)	6.332 (-0.027)	6.223 (-0.136)
100	3.097 (0.000)	3.410 (0.313)	3.057 (-0.041)	3.084 (-0.013)
110	0.502 (0.000)	0.597 (0.095)	0.437 (-0.066)	0.538 (0.036)
115	0.198 (0.000)	0.196 (-0.002)	0.212 (0.014)	0.215 (0.018)
125	0.032 (0.000)	0.014 (-0.019)	0.070 (0.037)	0.028 (-0.004)
130	0.014 (0.000)	0.003 (-0.011)	0.031 (0.017)	0.008 (-0.005)

(c) $T = \frac{1}{2}$

K	Exact (0.000)	BS (0.155)	D (0.044)	JR (0.079)
70	30.061 (0.000)	30.006 (-0.055)	30.057 (-0.004)	30.129 (0.068)
80	20.317 (0.000)	20.151 (-0.167)	20.387 (0.070)	20.407 (0.089)
85	15.662 (0.000)	15.491 (-0.171)	15.717 (0.054)	15.554 (-0.108)
95	7.478 (0.000)	7.686 (0.208)	7.431 (-0.047)	7.227 (-0.251)
100	4.457 (0.000)	4.899 (0.443)	4.434 (-0.023)	4.444 (-0.012)
110	1.373 (0.000)	1.615 (0.242)	1.275 (-0.099)	1.512 (0.139)
125	0.239 (0.000)	0.188 (-0.051)	0.282 (0.043)	0.261 (0.022)
135	0.079 (0.000)	0.034 (-0.045)	0.126 (0.047)	0.066 (-0.013)
150	0.017 (0.000)	0.002 (-0.015)	0.023 (0.007)	0.006 (-0.011)

(d) $T = 1$

K	Exact (0.000)	BS (0.224)	D (0.067)	JR (0.368)
65	35.170 (0.000)	35.038 (-0.132)	35.213 (0.043)	35.762 (0.592)
75	25.578 (0.000)	25.349 (-0.229)	25.691 (0.113)	25.577 (-0.001)
85	16.667 (0.000)	16.619 (-0.048)	16.639 (-0.028)	15.574 (-1.093)
95	9.217 (0.000)	9.719 (0.502)	9.118 (-0.099)	8.444 (-0.773)
100	6.408 (0.000)	7.114 (0.706)	6.338 (-0.071)	6.173 (-0.235)
120	1.433 (0.000)	1.561 (0.128)	1.344 (-0.090)	1.890 (0.456)
140	0.383 (0.000)	0.242 (-0.140)	0.488 (0.105)	0.475 (0.092)
160	0.119 (0.000)	0.030 (-0.090)	0.171 (0.052)	0.083 (-0.037)
180	0.042 (0.000)	0.003 (-0.039)	0.042 (0.000)	0.011 (-0.031)

Table 5.28.: Exact and approximated option prices with errors for different strikes K and maturities T in the NIG-CIR model for parameters as in Table 5.25 (risk-neutral measure): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **JR** refers to the approximation by [JR82], and **D** to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

(a) $T = 2$

K	Exact (0.000)	BS (0.320)	D (0.142)	JR (2.339)
55	45.276 (0.000)	45.076 (-0.200)	45.380 (0.104)	50.190 (4.914)
65	35.732 (0.000)	35.445 (-0.288)	35.928 (0.196)	35.290 (-0.443)
80	22.504 (0.000)	22.572 (0.068)	22.422 (-0.082)	15.116 (-7.388)
94	12.534 (0.000)	13.413 (0.878)	12.292 (-0.242)	8.217 (-4.317)
115	4.472 (0.000)	5.273 (0.801)	4.225 (-0.247)	6.209 (1.737)
140	1.548 (0.000)	1.464 (-0.085)	1.614 (0.065)	3.319 (1.771)
170	0.556 (0.000)	0.273 (-0.282)	0.788 (0.233)	0.922 (0.366)
200	0.238 (0.000)	0.048 (-0.190)	0.327 (0.090)	0.192 (-0.046)
240	0.092 (0.000)	0.005 (-0.088)	0.074 (-0.019)	0.020 (-0.072)

(b) $T = 4$

K	Exact (0.000)	BS (0.506)	D (0.242)	JR (80.056)
50	50.720 (0.000)	50.403 (-0.317)	50.971 (0.250)	5.492 (-45.23)
60	41.566 (0.000)	41.312 (-0.254)	41.752 (0.186)	-253.92 (-295.5)
80	25.430 (0.000)	26.053 (0.622)	25.135 (-0.295)	-235.14 (-260.6)
104	12.262 (0.000)	13.795 (1.533)	11.838 (-0.424)	-12.855 (-25.12)
135	4.941 (0.000)	5.637 (0.696)	4.675 (-0.266)	59.518 (54.58)
170	2.194 (0.000)	1.978 (-0.216)	2.432 (0.238)	34.969 (32.78)
230	0.785 (0.000)	0.333 (-0.452)	1.151 (0.367)	6.723 (5.938)
290	0.365 (0.000)	0.061 (-0.304)	0.471 (0.106)	1.110 (0.745)
370	0.165 (0.000)	0.007 (-0.158)	0.119 (-0.046)	0.108 (-0.058)

(c) $T = 8$

K	Exact (0.000)	BS (0.635)	D (0.243)	JR (323.0)
40	61.100 (0.000)	60.803 (-0.297)	61.318 (0.218)	1607 (1546)
60	44.175 (0.000)	44.368 (0.193)	44.099 (-0.076)	1120 (1076)
90	25.146 (0.000)	26.689 (1.543)	24.707 (-0.439)	172.35 (147.2)
126	12.704 (0.000)	14.366 (1.662)	12.259 (-0.445)	-54.944 (-67.65)
180	5.493 (0.000)	5.866 (0.373)	5.446 (-0.047)	-36.820 (-42.31)
260	2.271 (0.000)	1.736 (-0.534)	2.750 (0.479)	-7.441 (-9.711)
380	0.928 (0.000)	0.351 (-0.578)	1.284 (0.356)	-0.564 (-1.493)
540	0.414 (0.000)	0.057 (-0.356)	0.453 (0.039)	0.006 (-0.408)
780	0.181 (0.000)	0.006 (-0.175)	0.097 (-0.083)	0.007 (-0.174)

(d) $T = 16$

K	Exact (0.000)	BS (1.196)	D (0.654)	JR (196.0)
40	65.792 (0.000)	63.314 (-2.478)	63.334 (-2.458)	900 (834)
70	41.722 (0.000)	44.533 (2.811)	43.195 (1.473)	-217.77 (-259.5)
120	23.234 (0.000)	26.005 (2.770)	23.722 (0.488)	-146.80 (-170.0)
200	11.785 (0.000)	12.408 (0.623)	11.162 (-0.623)	-32.806 (-44.59)
330	5.180 (0.000)	4.655 (-0.525)	5.159 (-0.021)	-2.307 (-7.487)
560	1.885 (0.000)	1.218 (-0.667)	2.350 (0.465)	0.778 (-1.107)
930	0.712 (0.000)	0.245 (-0.466)	0.939 (0.227)	0.283 (-0.429)
1500	0.325 (0.000)	0.040 (-0.285)	0.292 (-0.033)	0.059 (-0.266)
2600	0.144 (0.000)	0.003 (-0.141)	0.049 (-0.095)	0.006 (-0.138)

Table 5.29.: Exact and approximated option prices with errors for different strikes K and maturities T in the NIG-CIR model for parameters as in Table 5.25 (risk-neutral measure): **Exact** refers to the exact option price, **BS** is the Black-Scholes price relative to volatility $\overline{\sigma}$, **JR** refers to the approximation by [JR82], and **D** to the approximation of this thesis. Values in brackets in the body of the table are differences of exact and approximated prices. Values in brackets in the head refer to the mean absolute error over strikes (MAEOS) of the respective approximation.

5.6. Approximate option pricing formulas in the literature

In Sections 5.6.1–5.6.7, we review some contributions on approximate option pricing that we consider most related to our work and that we use in our numerical studies. In Section 5.6.8, we give some further references.

5.6.1. Jarrow & Rudd (1982)

[JR82] is the earliest reference on approximate option pricing that we are aware of. The authors consider a European call option with maturity $T > 0$ and strike $K > 0$ in a quite general framework for the stock price. Their approach is based on the approximation of the risk-neutral density g of the terminal stock price S_T by means of Edgeworth techniques, using the density a of a suitable log-normal distribution as zero-order approximation. This analysis starts at the level of cumulant generating functions. Denoting by $\kappa^g(z)$ and $\kappa^a(z)$ the cumulant generating functions of g and the appropriate log-normal density a, the authors consider the formal expansions

$$\begin{aligned}\kappa^g(z) &= \sum_{k=1}^{N} \frac{1}{k!}\kappa_k^g z^k + o(z^N),\\ \kappa^a(z) &= \sum_{k=1}^{N} \frac{1}{k!}\kappa_k^a z^k + o(z^N),\end{aligned}$$

where $\kappa_k^g = (\kappa^g)^{(k)}(0)$ is the k-th cumulant of g, and analogously for κ^a. It is then concluded that

$$\kappa^g(z) = \kappa^a(a) + \sum_{k=1}^{N} \frac{1}{k!}\left(\kappa_k^g - \kappa_k^a\right) z^k + o(z^N).$$

The characteristic function of g is hence given by

$$e^{\kappa^g(iu)} = e^{\kappa^a(iu)} \left(1 + \sum_{k=1}^{N} \frac{1}{k!} E_k (iu)^k + o(u^N)\right),$$

where E_k, $k = 1, \dots, N$, are coefficients – depending on $\kappa_1^g, \dots, \kappa_N^g$ and $\kappa_1^a, \dots, \kappa_N^a$ – that result from a Taylor expansion and suitable reordering of $\exp\left(\sum_{k=1}^{N} \frac{1}{k!}\left(\kappa_k^g - \kappa_k^a\right) z^k\right)$. The authors apply Fourier inversion to the characteristic function of g to go back to the level of the density:

$$\begin{aligned}g(s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isu} e^{\kappa^g(iu)}\,\mathrm{d}u\\ &\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isu} e^{\kappa^a(iu)} \left(1 + \sum_{k=1}^{N} \frac{1}{k!} E_k (iu)^k\right) \mathrm{d}u\\ &= a(s) + \sum_{k=1}^{N} \frac{1}{2\pi} \frac{E_k}{k!} \int_{-\infty}^{\infty} e^{-isu} e^{\kappa^a(iu)} (iu)^k\\ &= a(s) + \sum_{k=1}^{N} \frac{E_k}{k!} (-1)^k a^{(k)}(s).\end{aligned}$$

This formal approximation to g is then inserted into the representation

$$c := e^{-rT} \int_{-\infty}^{\infty} (s-K)^+ g(s)\,\mathrm{d}s$$

of the option price. After further simplification, c is approximately given by a Black-Scholes price plus corrections that depend on derivatives of a at K and cumulants of g and a. The authors consider an expansion up to the fourth order. The first moment of the approximating log-normal distribution is fixed by the martingale condition on $e^{-rt}S_t$. In our numerical experiments, we follow "method 2" proposed by the authors and choose the variance of the normal distribution taken to the exponential such that it equals $\mathrm{Var}(\log(S_T))$. This corresponds to our zero-order approximation in geometric Lévy models, but it differs when our process V is not deterministic.

The analysis is formal; in particular, the authors do not provide conditions when the series related to their approach converges or quantify the error of the approximation analytically. One can check that the expressions in the approximation besides the cumulants can be written as cash greeks of the call option in the Black-Scholes model. We observe in our numerical experiments from Section 5.5 that the approximation by [JR82] yields unreasonable results for longer maturities.

We finally remark that if one considers the representation

$$c = e^{-rT} \int_{-\infty}^{\infty} (e^x - K)^+ \widetilde{g}(s)\,\mathrm{d}s$$

of the price instead and approximates the density $\widetilde{g}$ of the logarithmic terminal stock price $\log(S_T)$ by a normal distribution in the same spirit, the resulting fourth-order approximation coincides with ours in the case of geometric Lévy models, where we obtain reasonable results in particular for long maturities.

5.6.2. Hull & White (1987) / Ball & Roma (1994)

[HW87] consider bivariate stochastic volatility diffusion models, where the price process of the stock relative to the risk-neutral measure is given by

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sqrt{y_t}S_t\,\mathrm{d}W_t^1, \quad S_0 > 0, \\ \mathrm{d}y_t &= \kappa y_t\,\mathrm{d}t + \theta y_t\,\mathrm{d}W_t^2, \quad y_0 > 0, \end{aligned}$$

where $r \in \mathbb{R}$, $\kappa, \theta > 0$, and W^1, W^2 are independent standard Brownian motions, which is a crucial assumption. The key observation of the authors is that the price of a European option with payoff $f(S_T)$ at maturity $T > 0$ given by $c := e^{-rT}\mathrm{E}(f(S_T))$ can – by a conditioning argument with respect to y – be written in the form

$$c = \mathrm{E}\left(\mathrm{BS}(\overline{V}_T)\right), \tag{5.46}$$

where $\overline{V}_T := \frac{1}{T}\int_0^T y_s\,\mathrm{d}s$, and $\mathrm{BS}(\sigma^2)$ is the price of the option with payoff function f at maturity $T > 0$ in a Black-Scholes model with interest rate r, dependent on the squared volatility parameter σ. Basing on (5.46), the authors consider the formal power series

$$c = e^{-rT}\left\{\mathrm{BS}\left(\overline{\overline{V}}_T\right) + \frac{1}{2}\mathrm{BS}''\left(\overline{\overline{V}}_T\right)\mathrm{E}\left(\left(\overline{V}_T - \overline{\overline{V}}_T\right)^2\right) + \frac{1}{6}\mathrm{BS}'''\left(\overline{\overline{V}}_T\right)\mathrm{E}\left(\left(\overline{V}_T - \overline{\overline{V}}_T\right)^3\right)\right\} + \dots,$$

where $\overline{\overline{V}}_T := \mathrm{E}\left(\overline{V}_T\right)$. It is not made precise whether or under what conditions this series converges. Greeks – as sensitivities relative to squared volatility – and moments of the integrated squared volatility process appear in a natural way in the approximation. The authors do not compute the required moments of $\overline{V}_T - \overline{\overline{V}}_T$ in closed form for other volatility specifications than geometric Brownian motion but refer to numerical approaches to do so. [BR94] complement the study of [HW87] in this respect and compute the required moments when y follows a square root process, i.e., in the Heston model without correlation. They also consider the Stein & Stein model but evaluate the required moments by numerical differentiation of the moment generating function of $\overline{V}_T$ in their examples.

As mentioned in Remark 5.4.5, this power series approach (when taken up to the second order) is a special case of our approximation for bivariate diffusion models without correlation.

5.6.3. Fouque et al. (2000)

The contribution [FPS00] is very popular in the field of approximate techniques in Mathematical Finance. The authors consider approximations to option pricing in bivariate stochastic volatility models when the parameters controlling mean reversion speed and volatility of volatility are large. They exemplify their approach in a framework where stochastic volatility is driven by a Gaussian Ornstein-Uhlenbeck process, i.e., when the stock price process relative to the risk-neutral measure is given by

$$\begin{aligned} \mathrm{d}S_t^\varepsilon &= rS_t^\varepsilon\,\mathrm{d}t + g(y_t^\varepsilon)S_t^\varepsilon\,\mathrm{d}W_t^1, \quad S_0^\varepsilon = S_0, \\ \mathrm{d}y_t^\varepsilon &= \frac{1}{\varepsilon}(\eta - y_t^\varepsilon)\,\mathrm{d}t + \frac{\theta}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^2, \quad y_0^\varepsilon = y_0, \end{aligned}$$

making the dependence on $\varepsilon > 0$ explicit. Moreover, $r \in \mathbb{R}$, η, $\theta > 0$, $g(y)$ is a suitable function, and W^1 and W^2 are standard Brownian motions with $\mathrm{d}\langle W^1, W^2\rangle_t = \rho\,\mathrm{d}t$ for $\rho \in (-1,1)$.

The authors' aim is a first-order approximation to option prices as ε goes to 0. Note that, since also the volatility of volatility depends on ε, the approach is actually not a mere perturbation with respect to mean reversion speed, as it is often stated.

The analysis of [FPS00] is based on the partial differential equation (PDE) of the pricing function

$$P^\varepsilon(t,s,y) := e^{-r(T-t)}\mathrm{E}\left(f(S_T^\varepsilon)\,|\,S_t^\varepsilon = s, y_t^\varepsilon = y\right),$$

where f is the payoff function of a European option with maturity $T > 0$. The authors observe that the related pricing PDE is given by

$$\left(\frac{1}{\varepsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathscr{L}_1 + \mathscr{L}_2\right)P^\varepsilon = 0, \tag{5.47}$$

where $\mathscr{L}_0$, $\mathscr{L}_1$, $\mathscr{L}_2$ are explicitly known differential operators independent of ε. Then, the formal expansion

$$P^\varepsilon = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon^{\frac{3}{2}}P_3 + \ldots$$

is inserted into (5.47), which yields

$$\begin{aligned}\frac{1}{\varepsilon}\mathscr{L}_0P_0 + \frac{1}{\sqrt{\varepsilon}}(\mathscr{L}_0P_1 + \mathscr{L}_1P_0)\\ + (\mathscr{L}_0P_2 + \mathscr{L}_1P_1 + \mathscr{L}_2P_0)\\ + \sqrt{\varepsilon}(\mathscr{L}_0P_3 + \mathscr{L}_1P_2 + \mathscr{L}_2P_1)\\ + \dots\\ = 0.\end{aligned}$$

Hence, the differential equations $\mathscr{L}_0P_0 = 0$, $\mathscr{L}_0P_1 + \mathscr{L}_1P_0 = 0$, ... have to be satisfied simultaneously. A corresponding, involved analysis shows that $P_0(t,s)$ is the Black-Scholes pricing function of the option with payoff function f at maturity T relative to the volatility parameter

$$\overline{\sigma} := \sqrt{\int_{-\infty}^{\infty} g(y)^2 \varphi_{\eta,\frac{\theta^2}{2}}(y)\,\mathrm{d}y},$$

where $\varphi_{\eta,\frac{\theta^2}{2}}$ is the density of a standard distribution with mean η and variance $\frac{\theta^2}{2}$, which is the stationary distribution of y^1. Moreover,

$$P_1(t,s) = -(T-t)\left(V_2 s^2 \frac{\partial^2}{\partial s^2}P_0(t,s) + V_3 s^3 \frac{\partial^3}{\partial s^3}P_0(t,s)\right),$$

i.e., a linear combination of cash greeks of the Black-Scholes price P_0, where V_2, V_3 can be computed explicitly in terms of the model parameters. In the case $\rho = 0$, we have $V_3 = 0$.

Since the volatility parameter $\overline{\sigma}$ used for P_0 is related to the stationary distribution of y, the approximation is designed for maturities that are long in comparison to the mean reversion time of the volatility process, as also our numerical experiments in Section 5.5.3.4 indicate.

Due to the use of PDE techniques, also some exotic like barrier options can be treated by the approach of [FPS00]. Since the arguments for the first-order approximation are already involved, a generalization to higher orders seems complicated.

In the monograph [FPS00], only smooth payoff functions are considered. A verification that the approximation is indeed of first order for European call options is made precise in [FPSS03].

5.6.4. Alòs (2006)

In [Alò06], the author considers the pricing of a European option with continuous payoff function $f : \mathbb{R}_+ \to \mathbb{R}_+$ and maturity $T > 0$ in a stochastic volatility model with correlation, where the stock price process relative to the risk-neutral measure is given by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + y_tS_t(\rho\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2), \quad S_0 > 0.$$

There, W^1, W^2 are independent standard Brownian motions, $r \in \mathbb{R}$, $\rho \in [-1,1]$, and y is a non-negative, not necessarily Markovian process adapted to the filtration generated by W^1. Subject to regularity conditions on y, a decomposition of the option price $c := e^{-rT}\mathrm{E}(f(S_T))$ is derived by Malliavin techniques, given by

$$c = \mathrm{E}\left(\mathrm{BS}(0, \log(S_0), \sqrt{v_0})\right) + \frac{\rho}{2}\mathrm{E}\left(\int_0^T e^{-rt} H(t, \log(S_t), \sqrt{v_t})\Lambda_t \,\mathrm{d}t\right), \tag{5.48}$$

where $\mathrm{BS}(t,x,\sigma)$ denotes the price of the option with payoff function f and maturity T at point in time $t \in [0,T]$ and current logarithmic stock price $x \in \mathbb{R}$ in a Black-Scholes model with interest rate r and volatility parameter σ. Moreover, $v_t := \frac{1}{T-t}\int_t^T y_s^2 \,\mathrm{d}s$, $H(t,x,\sigma) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right)\mathrm{BS}(t,x,\sigma)$, and the process Λ is related to the Malliavin derivative of y_t, $t \in [0,T]$. Hence, (5.48) is a generalization of the approach from [HW87], cf. Section 5.6.2, to the case with correlation. Based on (5.48), the author proposes to use

$$c_{\text{approx}} := \mathrm{BS}\left(0, \log(S_0), \sqrt{\bar{v}_0}\right) + \frac{\rho}{2} H\left(0, \log(S_0), \sqrt{\bar{v}_0}\right)\mathrm{E}\left(\int_0^T \Lambda_t \,\mathrm{d}t\right) \tag{5.49}$$

with $\bar{v}_0 := \mathrm{E}\left(\frac{1}{T}\int_0^T y_t^2 \,\mathrm{d}t\right)$ as approximation to the option price. The zero-order Black-Scholes price coincides with ours only in the case of zero correlation.

If y follows a Gaussian Ornstein-Uhlenbeck process with the parametrization

$$\mathrm{d}y_t = \kappa(\eta - y_t)\,\mathrm{d}t + \theta\sqrt{\kappa}\,\mathrm{d}W_t^1, \quad y_0 > 0,$$

or a suitable transformation of such an Ornstein-Uhlenbeck process, it is shown that

$$\left|c - c_{\text{approx}}\right| \leq C\frac{\theta^2}{\kappa}(1 + |\log(\kappa)|)$$

for a constant $C > 0$. Hence, the proposed formula can be seen as an approximation of at least first order with respect to volatility of volatility θ. The approach is not based on perturbation in the classical sense, and it is not clear how to obtain higher-order approximations. E.g., in the case that y follows a Gaussian Ornstein-Uhlenbeck process, the term $\mathrm{E}\left(\int_0^T \Lambda_t \,\mathrm{d}t\right)$ can be computed explicitly in terms of the model parameters. The Black-Scholes greeks appearing in (5.48) and (5.49) result from the application of an Itō-type formula from Malliavin calculus to the function BS. In the related study [Alò12], cf. Section 5.6.7, the author notes that the required regularity conditions for y are "not trivial" to check in the Heston model.

5.6.5. Benhamou et al. (2010) / Lewis (2000)

[BGM10b] consider the approximate pricing of a European put option with strike $K > 0$ and maturity $T > 0$ in a Heston model with time-dependent parameters, where the logarithm X of the stock price process relative to the risk-neutral measure is given by

$$\begin{aligned}
\mathrm{d}X_t &= r\,\mathrm{d}t - \frac{1}{2}y_t\,\mathrm{d}t + \sqrt{y_t}\,\mathrm{d}W_t^1, \quad X_0 = \log(S_0), \\
\mathrm{d}y_t &= \kappa(\eta_t - y_t)\,\mathrm{d}t + \theta_t\sqrt{y_t}\,\mathrm{d}W_t^2, \quad y_0 > 0, \\
\mathrm{d}\langle W^1, W^2\rangle_t &= \rho_t\,\mathrm{d}t.
\end{aligned}$$

Here, S_0, κ, $r > 0$, $t \mapsto \eta_t$, $t \mapsto \theta_t$, and $t \mapsto \rho_t$ are deterministic and bounded functions on $[0,T]$, and W^1, W^2 are standard Brownian motions. In general, no fast exact method is known to compute option prices in this situation, which is why the authors aim at a fast to evaluate approximation. To this end, they parallel our general perturbation approach from Chapter 2 and connect the logarithmic stock price process of interest with the one of a time-dependent Black-Scholes model by introducing an artificial parameter $\varepsilon \in [0,1]$. More precisely, for $\varepsilon \in [0,1]$, they consider the family of processes given by

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= r\,\mathrm{d}t - \frac{1}{2}y_t^\varepsilon\,\mathrm{d}t + \sqrt{y_t^\varepsilon}\,\mathrm{d}W_t^1, \quad X_0^\varepsilon = \log(S_0), \\ \mathrm{d}y_t^\varepsilon &= \kappa(\eta_t - y_t^\varepsilon)\,\mathrm{d}t + \varepsilon\theta_t\sqrt{y_t^\varepsilon}\,\mathrm{d}W_t^2, \quad y_0^\varepsilon = y_0. \end{aligned}$$

The price of the put option relative to the logarithmic stock price X^ε, $\varepsilon \in [0,1]$, is hence given by

$$g(\varepsilon) := e^{-rT}\mathrm{E}\left(\left(K - e^{X_T^\varepsilon}\right)^+\right).$$

The first step to obtain a candidate for an approximation is the application of a conditioning argument on y^ε to conclude that

$$g(\varepsilon) = \mathrm{E}\left(\mathrm{BS}\left(X_0 + rT - \frac{1}{2}\int_0^T \rho_t^2 y_t^\varepsilon\,\mathrm{d}t + \int_0^T \rho_t\sqrt{y_t^\varepsilon}\,\mathrm{d}W_t^2, \int_0^T (1-\rho_t)^2 y_t^\varepsilon\,\mathrm{d}t\right)\right), \tag{5.50}$$

where $\mathrm{BS}(x, \sigma^2 T)$ is the price of the put option with maturity T in a Black-Scholes model with volatility parameter σ and initial logarithmic stock price $x \in \mathbb{R}$. The second ingredient is given by the first two derivatives (in L^p-sense) of y_t^ε relative to ε. Setting $y_{1,t}^\varepsilon := \frac{\partial}{\partial\varepsilon}y_t^\varepsilon$ and $y_{2,t}^\varepsilon := \frac{\partial^2}{\partial\varepsilon^2}y_t^\varepsilon$, the authors state the corresponding stochastic differential equations for $y_{1,\cdot}^\varepsilon$ and $y_{2,\cdot}^\varepsilon$. In particular, they obtain in the case $\varepsilon = 0$

$$\begin{aligned} y_t^0 &= e^{-\kappa t}\left(y_0 + \int_0^t \kappa e^{\kappa s}\eta_s\,\mathrm{d}s\right), \\ y_{1,t}^0 &= e^{-\kappa t}\int_0^t e^{\kappa s}\theta_s\sqrt{y_s^0}\,\mathrm{d}W_s^2, \\ y_{2,t}^0 &= e^{-\kappa t}\int_0^t e^{\kappa s}\theta_s\frac{y_{1,s}^0}{\sqrt{y_s^0}}\,\mathrm{d}W_s^2. \end{aligned} \tag{5.51}$$

Based on (5.51), the authors consider a formal second-order Taylor expansion of the random variable in the expectation from (5.50) around $\varepsilon = 0$ evaluated at $\varepsilon = 1$, which incorporates the derivatives of $\mathrm{BS}(\cdot,\cdot)$, i.e., greeks, up to the second order. The expectation of this expansion is proposed as approximation to $g(1)$. It remains unclear whether this really corresponds to a second-order expansion around $\varepsilon = 0$ of the real-valued function g itself.

After further computations, the proposed second-order approximation turns out to be of the form

$$g(\varepsilon) \approx \mathrm{BS}(\log(S_0), \sigma^2 T) + \sum_{i=1}^{2} D_{1^i 2} a_i \mathrm{BS}(\log(S_0), \sigma^2 T) + \sum_{i=0}^{1} b_i D_{1^{2i} 2^2}\mathrm{BS}(\log(S_0), \sigma^2 T),$$

i.e., it is given by a Black-Scholes price plus corrections involving greeks in this Black-Scholes model. There, $\sigma^2 := \mathrm{E}\left(\frac{1}{T}\int_0^T y_t\,\mathrm{d}t\right)$. Hence, the zero-order term corresponds to ours only in the case of zero correlation. The coefficients a_1, a_2, b_0, and b_2 are given by iterated integrals of the functions η, θ, and ρ. They can be computed explicitly in the case of constant parameters. The resulting approximation is used for comparison in our numerical experiments in Section 5.5.2. In the case of constant parameters, the approximation of [BGM10b] coincides with the one by [Lew00, Chapter 3], who performs a classical perturbation approach in Heston-like stochastic volatility models with respect to volatility of volatility by analyzing the PDE for the pricing function.

[BGM10b] further show that the error of their approximation is in $O\left(T^2\left(\sup_{t\in[0,T]}|\xi_t|\right)^3\right)$. Hence, their approximation can be understood to be of second order with respect to the volatility of volatility function.

The approach of [BGM10b] is based on Malliavin techniques. Generalizations to higher orders seem possible, while the concrete computations are probably quite involved.

5.6.6. Fukasawa (2011)

[Fuk11b] considers a very abstract framework, in which several perturbation approaches from the literature, e.g., the one of Section 5.6.3, can be embedded. The author considers a sequence X^n, $n \in \mathbb{N}$, of logarithmic stock price processes under the risk-neutral measure of the form

$$X^n = \log(S_0) + A^n + U^n + \int_0^\cdot g_t^n \,\mathrm{d}W_t + \int_0^\cdot h_t^n \,\mathrm{d}C_t^n,$$

where A^n is a suitable exponential compensator to guarantee the martingale property of e^{X^n}, U^n is a continuous martingale, g^n and h^n are adapted integrands, and

$$C_t^n = \varepsilon_n \sum_{j=1}^{N_{\Lambda_t^n}} Z_j.$$

Here, N is a standard Poisson process, $Z_1, Z_2, \ldots$ are random variables independent of N, and Λ^n is an increasing and continuous process. Moreover, ε_n, $n \in \mathbb{N}$, is a deterministic positive sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, for fixed $n \in \mathbb{N}$, C^n is a time-changed compound Poisson process.

The aim of [Fuk11b] is to derive a first-order approximation with respect to ε_n of the price a European option with payoff function f at maturity $T > 0$ around a Black-Scholes price. Intuitively, ε_n, $n \in \mathbb{N}$, can be understood as the rate at which the non-Gaussian features of X_n must vanish at least. If this is satisfied, a first-order approximation to the density of X^n is derived building on a general result from Malliavin calculus. The zero-order term is the density of the normal distribution, and the first-order correction depends on the distributional limit of quantities related to $[M^n, M^n]_T$ and $\langle M^n, M^n\rangle_T$, where M^n denotes the martingale part of X^n. In applications, the corrections have to be computed depending on the specific setup of X^n.

The author exemplifies his general framework for some specific situations, including a jump-diffusion model in Section 3.1. There, he sets $U^n \equiv 0$, $g^n \equiv 0$, $h^n \equiv 1$, and $\Lambda_t^n = \frac{1}{\varepsilon_n^2} t$. If the random variables $Z_1, Z_2, \ldots$ are normally distributed, one obtains the Merton model with normal jumps. The corresponding approximation by [Fuk11b] is used in the numerical tests in Section 5.5.4. Note that the resulting sequence C^n of compound Poisson processes corresponds to our rescaling of the Lévy process L. While the zero-order terms of both approximations coincide in this case, the first-order approximation of [Fuk11b], however, takes only the third moment of C into account, while we incorporate also the more important fourth moment because we work with a second-order approximation. Since the result from Malliavin calculus underlying the approach of [Fuk11b] is only available for first-order expansions, generalization to higher orders is left to future research and not obvious.

[Fuk11b] parallels our approach also in the respect that he does not perturb a specific model with respect to a univariate parameter but studies a sequence of models – in our language a curve to geometric Brownian motion. However, while we propose concrete curves to obtain explicit formulas, the author aims at an abstract analysis. Moreover, up to Edgeworth-based contributions, [Fuk11b] is the only reference we are aware of that deals with approximate option pricing around Black-Scholes in models with jumps as well.

5.6.7. Alòs (2012)

The contribution [Alò12] is similar in spirit to [Alò06]. The author focuses explicitly on the Heston model, where the price process of the stock relative to the risk-neutral measure is given by

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sqrt{y_t}S_t\left(\rho\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2\right), \quad S_0 > 0, \\ \mathrm{d}y_t &= \kappa(\eta - y_t)\,\mathrm{d}t + \theta\sqrt{y_t}\,\mathrm{d}W_t^1, \quad y_0 > 0, \end{aligned}$$

for $r \in \mathbb{R}$, $\kappa, \eta, \theta > 0$. By mere Itō calculus, a decomposition formula for the price $c := e^{-rT}\mathrm{E}\left(f(S_T)\right)$ of a European option with payoff function f and maturity $T > 0$ is derived, given by

$$\begin{aligned} c = {} & \mathrm{BS}\left(0, \log(S_0), \sqrt{v_0}\right) + \frac{\rho}{2}\mathrm{E}\left(\int_0^T e^{-rt} H(t, \log(S_t), \sqrt{v_t})\sqrt{y_t}\,\mathrm{d}\langle M, W^1\rangle_t\right) \\ & + \frac{1}{8}\mathrm{E}\left(\int_0^T e^{-rt} K(t, \log(S_t), \sqrt{v_t})\,\mathrm{d}\langle M, M\rangle_t\right), \end{aligned}$$

where $\mathrm{BS}(t,x,\sigma)$ denotes the price of the option with payoff function f at maturity T in the Black-Scholes model with interest rate r and volatility parameter σ if the logarithmic stock price at time $t \in [0,T]$ is given by $x \in \mathbb{R}$. Moreover, $v_t := \frac{1}{T-t}\int_t^T \mathrm{E}\left(y_s | \mathscr{F}_t\right)\mathrm{d}s$, $M_t := \int_0^T \mathrm{E}\left(y_s | \mathscr{F}_t\right)\mathrm{d}s$,

$$H(t,x,\sigma) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right)\mathrm{BS}(t,x,\sigma), \quad \text{and} \quad K(t,x,\sigma) := \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right)\mathrm{BS}(t,x,\sigma).$$

Based on this decomposition, the author suggests to use

$$\begin{aligned} c_{\mathrm{approx}} := {} & \mathrm{BS}\left(0, \log(S_0), \sqrt{v_0}\right) + \frac{\rho}{2}H\left(0, \log(S_0), \sqrt{v_0}\right)\mathrm{E}\left(\int_0^T \sqrt{y_t}\,\mathrm{d}\langle M, W^1\rangle_t\right) \\ & + \frac{1}{8}K\left(0, \log(S_0), \sqrt{v_0}\right)\mathrm{E}\left(\int_0^T \mathrm{d}\langle M, M\rangle_t\right) \end{aligned} \tag{5.52}$$

as approximation to the option price. As for [Alò06], the zero-order Black-Scholes price coincides with ours only in the case of zero correlation.

Based on several technical lemmas that explicitly use that y is a square root process, it is shown that the approximation error $\left|c - c_{\text{approx}}\right|$ goes to 0 when θ or T do, while the corresponding rates delicately depend on the parameters κ, η, and θ. The expectations in (5.52) can be computed explicitly in terms of the model parameters. The corresponding formulas in [Alò12, Remark 3.8] seem to be incorrect since we were only able to reproduce the numbers from the numerical illustration in [Alò12, Section 4] by using the expressions we computed independently.

As in [Alò06], the approach is not based on classical perturbation; rather, an approximation is proposed, and then a corresponding error estimate is given. Generalization to higher orders is not obvious.

5.6.8. Further references

The literature on approximate option pricing is very vast. Let us provide some further references, which are of course not to be considered as comprehensive. In their studies [BGM09] and [BGM10a] related to [BGM10b] discussed in Section 5.6.5, the authors derive approximations to prices by perturbing local volatility models with and without jumps around the Merton model resp. the Black-Scholes model. [PPR13] have a similar aim but base their approach on PDE techniques. [AS09] expand the price of a European option in bivariate stochastic volatility models relative to the correlation parameter. [Fuk11a] reviews the approximation by [FPS00] in the light of Edgeworth expansions. [HW99] obtain approximations to European option prices and implied volatilities in CEV-like models by a PDE approach. [GHL$^+$12] consider the asymptotics of implied volatility in local volatility models. [GHL$^+$12] expand prices in the Black-Scholes model relative to the volatility parameter.

5.7. Proofs

5.7.1. General framework and perturbation of stochastic volatility models

Lemma 5.7.1. *Let (S_0, L, V, U) be a stochastic volatility model as in Definition 5.2.1. Then, e^L is an intrinsic martingale.*

PROOF. Denoting by $\tilde{\kappa}$ the cumulant generating function of L, the equality $\mathrm{E}\left(e^{L_1}\right) = 1$ implies $\tilde{\kappa}(1) = 0$. By the representation of $\tilde{\kappa}$ in terms of its Lévy-Khintchine triplet according to [Sat99, Theorem 25.17], Remark E.0.5 and Proposition E.0.14, we see that $\tilde{\kappa}(1)I = 0$ is the exponential compensator of L. Hence, e^L is a local martingale relative to its intrinsic filtration. By [Kal00, Lemma 4.4(3)], e^L is then an intrinsic martingale. $\square$

PROOF OF THEOREM 5.2.2. At first, we work on the stochastic basis $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\in\mathbb{R}_+}, P)$ with $\mathscr{G} := \mathscr{G}_{\infty-} = \sigma(\cup_{s\geq 0}\mathscr{G}_s)$. Then, the process θ given by

$$\theta_t := \inf\{s \in \mathbb{R}_+ : V_s > t\}, \quad t \in \mathbb{R}_+,$$

is a time change (cf. [Jac79, Section X.1.§a]) because V is adapted to $\mathbf{G}$. Since $\exp(U)$ is a $\mathbf{G}$-martingale by assumption, we have $\exp(U) \in \mathscr{M}_{\mathrm{loc}}(\mathbf{G})$ by [JS03, Proposition I.1.47(a)]. Since all paths of V are strictly increasing by assumption, we have that

$$[\![\theta_{t-}, \theta_t]\!] = \emptyset \quad \text{for all } t \in \mathbb{R}_+.$$

Hence, every process, and in particular $\exp(U)$, is adapted to the time change θ in the sense of [Jac79, Définition 10.13]. By [Jac79, Théorème 10.16], the time-changed process $Y := \exp(U_\theta)$ in the sense of [Jac79, Equation 10.6] is in $\mathscr{M}^J_{\mathrm{loc}}(\mathbf{H})$, where the time-changed filtration $\mathbf{H} := (\mathscr{H}_t)_{t\in\mathbb{R}_+}$ is given by $\mathscr{H}_t = \mathscr{G}_{\theta_t}$, $t \in \mathbb{R}_+$,

$$J := \{(\omega, t) \in \Omega \times \mathbb{R}_+ : \theta_{t-}(\omega) < \infty\}$$

(cf. [Jac79, Equation 10.7]), and

$$\mathscr{M}^J_{\mathrm{loc}}(\mathbf{H}) = \left\{X \text{ stochastic process} : X^T \in \mathscr{M}_{\mathrm{loc}}(\mathbf{H}) \text{ for all } \mathbf{H}\text{-stopping times } T \text{ with } [\![0,T]\!] \subset J\right\}.$$

Now, we consider in addition the filtration $\mathbf{L} := (\mathscr{L}_t)_{t\in\mathbb{R}_+}$ given by $\mathscr{L}_t := \sigma(L_s : s \leq t)$ for $t \in \mathbb{R}_+$. Observe that $\mathbf{L}$ is independent of $\mathbf{H}$ since

$$\mathscr{H}_t = \mathscr{G}_{\theta_t} \subset \mathscr{G}_\infty = \mathscr{G}_{\infty-} \quad \text{for all } t \in \mathbb{R}_+,$$

and by assumption, $\mathscr{G}_{\infty-}$ and $\mathscr{L}_{\infty-}$ are independent, and hence $\mathscr{H}_{\infty-}$ and $\mathscr{L}_{\infty-}$ are independent as well.

From now on, consider the filtered probability space $(\Omega, \mathscr{J}, (\mathscr{J}_t)_{t\in\mathbb{R}_+}, P)$ with

$$\mathscr{J} := \sigma(\mathscr{H}_{\infty-} \cup \mathscr{L}_{\infty-}) \quad \text{and} \quad \mathscr{J}_t := \sigma(\mathscr{H}_t \cup \mathscr{L}_t), \quad t \in \mathbb{R}_+,$$

setting $\mathbf{J} := (\mathscr{J}_t)_{t\in\mathbb{R}_+}$. Then, for every $\mathbf{H}$-stopping time T such that $[\![0,T]\!] \subset J$, we have that $Y^T \exp(L) \in \mathscr{M}_{\mathrm{loc}}(\mathbf{J})$. To see this, fix such a T, and let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence of $\mathbf{H}$-stopping times such that $Y^{T\wedge\tau_n} \in \mathscr{M}(\mathbf{H})$, which is possible by the preceding reasoning. (Clearly, $(\tau_n)_{n\in\mathbb{N}}$ is then also a sequence of $\mathbf{J}$-stopping times such that $Y^{T\wedge\tau_n}$ is $\mathbf{J}$-adapted.) By Lemma A.0.2, $\exp(L)$ remains a martingale also with respect to the enlarged filtration $\mathbf{J}$ since $\mathscr{L}_{\infty-}$ is independent of $\mathscr{H}_{\infty-}$. We have that $\exp(L)Y^{T\wedge\tau_n}$ is a $\mathbf{J}$-martingale for all $n \in \mathbb{N}$ since for all $0 \leq s \leq t$

$$\begin{aligned}
\mathrm{E}\left(\exp(L_t)Y_t^{T\wedge\tau_n}\middle|\, \mathscr{J}_s\right) &= \mathrm{E}\left(\exp(L_t)Y_t^{T\wedge\tau_n}\middle|\, \sigma(\mathscr{H}_s \cup \mathscr{L}_s)\right) \\
&= \mathrm{E}\left(\mathrm{E}\left(\exp(L_t)Y_t^{T\wedge\tau_n}\middle|\, \sigma(\mathscr{H}_s \cup \mathscr{L}_t)\right)\middle|\, \sigma(\mathscr{J}_s)\right) \\
&= \mathrm{E}\left(\exp(L_t)\mathrm{E}\left(Y_t^{T\wedge\tau_n}\middle|\, \sigma(\mathscr{H}_s \cup \mathscr{L}_t)\right)\middle|\, \sigma(\mathscr{J}_s)\right) \\
&= \mathrm{E}\left(\exp(L_t)\mathrm{E}\left(Y_t^{T\wedge\tau_n}\middle|\, \mathscr{H}_s\right)\middle|\, \sigma(\mathscr{J}_s)\right) \\
&= \mathrm{E}\left(\exp(L_t)Y_s^{T\wedge\tau_n}\middle|\, \sigma(\mathscr{J}_s)\right) \\
&= \exp(L_s)Y_s^{T\wedge\tau_n}.
\end{aligned}$$

Here, the fourth equality sign is justified by [Bau78, Satz 54.4] since $\mathscr{H}_s \subseteq \mathscr{H}_t$ and $\mathscr{L}_t$ are independent and $Y_t^{T\wedge\tau_n}$ is $\mathscr{H}_t$-measurable. (Integrability of $\exp(L_t)Y_t^{T\wedge\tau_n}$ follows by setting $s = 0$ above.) We have that $\left(\exp(L)Y^{T\wedge\tau_n}\right)^{\tau_n} = (\exp(L)Y^T)^{\tau_n}$ is a $\mathbf{J}$-martingale as well since the class of martingales is stable under stopping. Hence, $\exp(L)Y^T \in \mathscr{M}_{\mathrm{loc}}(\mathbf{J})$, where $(\tau_n)_{n\in\mathbb{N}}$ is a corresponding localizing sequence.

We will now apply the time change V to the process $\exp(L)Y^T$ on the filtered probability space $(\Omega, \mathscr{J}, (\mathscr{J}_t)_{t\in\mathbb{R}_+}, P)$, where T is an arbitrary $\mathbf{H}$-stopping time such that $[\![0,T]\!] \subset J$. Since for all $t\in\mathbb{R}_+$ we noted that θ_t is a $\mathbf{G}$-stopping time, θ_t is $\mathscr{G}_{\theta_t}$- and hence $\mathscr{J}_t$-measurable by [JS03, I.1.14], i.e., θ is adapted to $\mathbf{J}$. Note that we have

$$V_t = \inf\{s\in\mathbb{R}_+ : \theta_s > t\} \quad \text{for all } t\in\mathbb{R}_+.$$

Hence, V is a time change (as right-inverse of an adapted process) on the filtered probability space $(\Omega, \mathscr{J}, (\mathscr{J}_t)_{t\in\mathbb{R}_+}, P)$. By the continuity of V, the process $\exp(L)Y^T$ is adapted to the time change V, and hence the time-changed process $\exp(L_V)Y_V^T$ is in $\mathscr{M}_{\mathrm{loc}}(\mathbf{N})$ by [Jac79, Théorème 10.16], where $\mathscr{N}_t := \mathscr{J}_{V_t}$, $t\in\mathbb{R}_+$, and $\mathbf{N} := (\mathscr{N}_t)_{t\in\mathbb{R}_+}$. For every $n\in\mathbb{N}$, V_n is an $\mathbf{H}$-stopping time because θ is adapted to $\mathbf{H}$. Moreover, obviously $[\![0,V_n]\!] \subset J$. Hence, $\exp(L_V)Y_V^{V_n} \in \mathscr{M}_{\mathrm{loc}}(\mathbf{N})$ for all $n\in\mathbb{N}$. Note that by the fact that V is strictly increasing $\theta_{V_t} = t$ for all $t\in\mathbb{R}_+$, and hence

$$\left(\exp(L_V)Y_V^{V_n}\right)_t = \exp(L_{V_t})\exp\left(U_{\theta_{V_t\wedge V_n}}\right) = \exp(L_{V_t})\exp\left(U_{t\wedge n}\right) \quad \text{for all } t\in\mathbb{R}_+.$$

Observe that for $t\in\mathbb{R}_+$

$$\begin{aligned}\mathrm{E}\left(\exp(L_{V_t})\exp\left(U_{t\wedge n}\right)\right) &= \mathrm{E}\left(\mathrm{E}\left(\exp\left(L_{V_t}+U_{t\wedge n}\right)\middle|\,\sigma(V_t,U_{t\wedge n})\right)\right)\\ &= \mathrm{E}\left(\exp(U_{t\wedge n})\mathrm{E}\left(\exp(L_{V_t})\middle|\,\sigma(V_t,U_{t\wedge n})\right)\right).\end{aligned}$$

By Lemmas A.0.4 and 5.7.1, we have that $\mathrm{E}\left(\exp(L_{V_t})\middle|\,\sigma(V_t,U_{t\wedge n})\right) = 1$ since $\exp(L)$ is a martingale independent of (U,V). Moreover, $\mathrm{E}\left(\exp(U_{t\wedge n})\right) = \mathrm{E}\left(\exp(U_0)\right) = 1$ since U is a martingale and $U_0 = 0$. Hence, $\exp(L_V)\exp\left(U^n\right)$ is a positive local $\mathbf{N}$-martingale with constant expectation and thus an $\mathbf{N}$-martingale by Lemma A.0.1 for all $n\in\mathbb{N}$. This implies that $\exp(L_V+U)$ is an $\mathbf{N}$-martingale.

To complete the proof, it remains to show that $\mathscr{R}_t \subseteq \mathscr{N}_t$ for all $t\in\mathbb{R}_+$, where $\mathbf{R} := (\mathscr{R}_t)_{t\in\mathbb{R}_+}$ denotes the filtration generated by (L_V,V,U). If this holds, $\exp(L_V+U)$ is obviously adapted to $\mathbf{R}$, and hence it is an $\mathbf{R}$-martingale by [Jac79, Proposition 9.14] (shrinkage of the filtration preserves the martingale property if adaptivity continues to hold). Fix now $t\in\mathbb{R}_+$. L is càdlàg and by construction adapted to $\mathbf{J}$, hence it is $\mathbf{J}$-optional. Since V_s is $\mathbb{R}_+$-valued and a $\mathbf{J}$-stopping time for all $0\le s\le t$, we have that L_{V_s} is $\mathscr{J}_{V_s} \subset \mathscr{J}_{V_t}$-measurable for all $s\le t$ by [JS03, Proposition I.1.21(a)]. This shows $\sigma(L_{V_s} : s\le t) \subset \mathscr{N}_t$. Since V_s is a $\mathbf{J}$-stopping time for all $s\le t$, we have by [JS03, I.1.14] that V_s is $\mathscr{J}_{V_s} = \mathscr{N}_s \subset \mathscr{N}_t$-measurable for all $s\le t$, which shows $\sigma(V_s : s\le t)\, subset \mathscr{N}_t$. By the preceding reasoning, we have for all $s\le t$ that $\exp(-L_{V_s})\exp(L_{V_s}+U_s) = \exp(U_s)$ in $\mathscr{N}_t$-measurable, which shows $\sigma(U_s : s\le t) \subseteq \mathscr{N}_t$ and completes the proof. □

PROOF OF LEMMA 5.2.6. Let us begin with (ii): by [CT03, Proposition 3.13], we have for all $\lambda\in(0,1]$ and $t\in\mathbb{R}_+$ $\mathrm{Var}\left(L_t^\lambda\right) = \mathrm{Var}\left(\lambda L_{\frac{t}{\lambda^2}}\right) = \lambda^2\frac{1}{\lambda^2}\mathrm{Var}(L_t) = \mathrm{Var}(L_t) = t\mathrm{Var}(L_1)$. (iii) is obvious in view of [CT03, Proposition 3.14] since $\lambda \in (0,1]$. The form of κ^λ in (iv) is clear by the definition of the cumulant generating function and L^λ in (5.8). Finally, L^λ is by definition again a Lévy process with càdlàg paths, and independence of (V,U) is obviously preserved. Note that by (iv), $\mathrm{E}\left(e^{L_1^\lambda}\right) = e^{\kappa^\lambda(1)} = 1$, and by (ii), $\mathrm{Var}\left(L_1^\lambda\right) > 0$, which completes the proof. □

PROOF OF LEMMA 5.2.7. Proposition 5.7.3 yields directly that

$$\lim_{\lambda\to 0} e^{\kappa^\lambda(y)} = e^{-\frac{1}{2}\mathrm{Var}(L_1)y+\frac{1}{2}\mathrm{Var}(L_1)y^2} \quad \text{for all } y \in i\mathbb{R}.$$

By Lévy's continuity theorem (cf., e.g., [Sat99, Proposition 2.5(vii)]), the univariate marginals of L^λ converge to the univariate marginals of $-\frac{1}{2}\mathrm{Var}(L_1)I + \sqrt{\mathrm{Var}(L_1)}B$ as $\lambda \to 0$, where B denotes standard Brownian motion. By [JS03, Corollary VII.3.6], this implies convergence of the whole process, which completes the proof. □

PROOF OF 5.2.10. The assertion is obvious for $\lambda = 1$. Hence, let $\lambda \in (0,1]$ in the remainder of the proof. By Proposition E.0.6, the semimartingale λM allows for differential characteristics $\left(b^{\lambda M}, c^{\lambda M}, F^{\lambda M}\right)$ relative to the same truncation function $h^M : \mathbb{R} \to \mathbb{R}$, and for all $t \in \mathbb{R}_+$

$$F_t^{\lambda M} = \int 1_G(\lambda x)\, F_t^M(\mathrm{d}x) = F_t^M\left(\frac{G}{\lambda}\right), \quad G \in \mathscr{B} \text{ with } 0 \notin G. \tag{5.53}$$

By Proposition E.0.11, the fact that λM is exponentially special is equivalent to

$$\int_0^t \int e^x 1_{\{x>1\}} F_s^{\lambda M}(\mathrm{d}x)\mathrm{d}s < \infty \quad \text{for all } t \in \mathbb{R}_+. \tag{5.54}$$

Using (5.53), we see that for all $t \in \mathbb{R}_+$

$$\int_0^t \int e^x 1_{\{x>1\}} F_s^{\lambda M}(\mathrm{d}x)\,\mathrm{d}s = \int_0^t \int e^{\lambda x} 1_{\{\lambda x>1\}}\, F_s^M(\mathrm{d}x)\,\mathrm{d}s \le \int_0^t \int e^x 1_{\{x>1\}} F_s^M(\mathrm{d}x)\,\mathrm{d}s < \infty.$$

The right-hand side is finite by Proposition E.0.11 and the fact that M is exponentially special. Hence, λM is exponentially special. The representation of the exponential compensator $K(\lambda M)$ of λM is directly given by Proposition E.0.14. □

PROOF OF PROPOSITION 5.2.12. For the remainder of the proof, consider an arbitrary but fixed $u \in \mathbb{R}_+$. We will show that for all $\lambda \in (0,1)$ the stopped process $\exp(U^\lambda)^u$ is a uniformly integrable martingale, which will yield the assertion (the cases $\lambda = 0$ and $\lambda = 1$ are trivial). By [KMK10, Lemma 2.3], the stopped process M^u allows as well for differential characteristics $\left(b^{M^u}, c^{M^u}, F^{M^u}\right) = \left(b^M 1_{[\![0,u]\!]}, c^M 1_{[\![0,u]\!]}, F^M 1_{[\![0,u]\!]}\right)$. By the same argument as in the proof of Lemma 5.2.10, we see that λM^u is exponentially special for all $\lambda \in (0,1)$, and for its exponential compensator $K(\lambda M^u)$ we have $K(\lambda M^u) = K(\lambda M)^u$. Hence, $(U^\lambda)^u = \lambda M^u - K(\lambda M^u)$. To show that $\exp(U^\lambda)^u$ is a uniformly integrable martingale, we use [KS02a, Theorem 3.6]. In order to employ it to U^λ for $\lambda \in (0,1)$, we have to show that there exists $\varepsilon > 0$ such that

$$\sup\{\mathrm{E}(\exp((1+\varepsilon)(\lambda M_S^u - K(\lambda M^u)_S))) : S \text{ finite stopping time }\} < \infty. \tag{5.55}$$

For given $\lambda \in (0,1)$, choose $\varepsilon = \frac{1}{\lambda} - 1 > 0$ (then $(1+\varepsilon)\lambda = 1$). Then, since $K(\lambda M^u) \geq 0$ by Lemma A.0.5, condition (5.55) holds if

$$\sup\left\{\mathrm{E}\left(\exp\left(M_S^u\right)\right) : S \text{ finite stopping time}\right\} < \infty. \tag{5.56}$$

By the assumption that M is a martingale, the integrability condition on $\exp(M)$, and Jensen's inequality, the process $\exp(M)$ is a submartingale. By Doob's optional stopping theorem (cf., e.g., [JS03, Theorem I.1.39(b)]) applied to bounded stopping times, we obtain

$$\mathrm{E}\left(\exp\left(M_S^u\right)\right) = \mathrm{E}\left(\exp\left(M_{u\wedge S}\right)\right) \leq \mathrm{E}\left(\exp\left(M_u\right)\right) < \infty$$

for all finite stopping times S, which shows (5.56) and completes the proof. □

PROOF OF LEMMA 5.2.18. Let us consider Assertion (i). In view of Assumption 5.2.15, $\mathrm{E}(V_t)$ is clearly real-valued for all $t \in \mathbb{R}_+$. Continuity follows from dominated convergence and the fact that V is by assumption a continuous, increasing process. To see that $t \mapsto \mathrm{E}(V_t)$ is strictly increasing, let $0 \leq s < t$. Then, $\mathrm{E}(V_t) - \mathrm{E}(V_s) = \mathrm{E}(V_t - V_s) \geq 0$ since $V_t - V_s > 0$ because V is strictly increasing. Moreover, we cannot have $\mathrm{E}(V_t) - \mathrm{E}(V_s) = 0$ since this would imply $V_t - V_s = 0$ almost surely, hence $t \mapsto \mathrm{E}(V_t)$ is strictly increasing. To treat Assertion (ii), we use that for all $t \in \mathbb{R}_+$ we have $\mathrm{Var}(M_t) = \mathrm{E}(\langle M,M\rangle_t)$ by Lemma 5.7.12 below. Since the predictable quadratic variation $\langle M,M\rangle$ is increasing by [JS03, Theorem I.4.2], monotonicity is obvious. By the same theorem, $\langle M,M\rangle$ is continuous because M is quasi-left-continuous by Assumption 5.2.16. The continuity of $t \mapsto \mathrm{Var}(M_t) = \mathrm{E}(\langle M,M\rangle_t)$ then follows as for $t \mapsto \mathrm{E}(V_t)$ by dominated convergence using that $\langle M,M\rangle$ is continuous and increasing. □

PROOF OF PROPOSITION 5.2.20. Assertion (i) is clear. Let us consider Assertion (ii). By definition of L^0, V^0 and U^0 from (5.9), (5.13) and (5.12), we have

$$\log\left(S_T^0\right) = \log(S_0) - \frac{1}{2}\mathrm{Var}(L_1)\left(\mathrm{E}(V_T) + \frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}\right) + \sqrt{\mathrm{Var}(L_1)}\,W_{\mathrm{E}(V_T)+\frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}}.$$

Taking into account the definition of $\overline{\sigma}$, the desired distributional property of S_T^0 follows by the fact that $W_t \sim N(0,t)$ for all $t \in \mathbb{R}_+$. Note that if the derivative in (5.17) exists, it is non-negative since $t \mapsto \mathrm{E}(V_t)$ and $t \mapsto \mathrm{Var}(M_t)$ are increasing by Lemma 5.2.18. The same then holds for $\widetilde{\sigma}(t) := \sqrt{\frac{\partial}{\partial t}\left(\mathrm{E}(V_t) + \frac{\mathrm{Var}(M_t)}{\mathrm{Var}(L_1)}\right)}$, $t \in \mathbb{R}_+$, and also $\lim_{t\to\infty}\int_0^t \widetilde{\sigma}^2(s)\,\mathrm{d}s = \infty$. By Proposition 5.3.2, there exists a standard Brownian motion $\overline{W}$ such that for all $t \in \mathbb{R}_+$

$$W_{\mathrm{E}(V_t)+\frac{\mathrm{Var}(M_t)}{\mathrm{Var}(L_1)}} = W_{\int_0^t \widetilde{\sigma}^2(s)\,\mathrm{d}s} = \int_0^t \widetilde{\sigma}(s)\,\mathrm{d}\overline{W}_s.$$

Since $\sigma(t) = \sqrt{\mathrm{Var}(L_1)}\,\widetilde{\sigma}(t)$, we finally obtain for all $t \in \mathbb{R}_+$

$$\log\left(S_t^0\right) = \log(S_0) - \frac{1}{2}\int_0^t \sigma^2(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\overline{W}_s,$$

which completes the proof. □

PROOF OF PROPOSITION 5.2.23. In the case $A \in [0,1]$, we have $A \in D$ since $\mathrm{E}\left(e^{L_1}\right) < \infty$ by assumption. Hence, $A \in D^\lambda$ for all $\lambda \in [0,1]$ by Lemma 5.2.6(iii). Moreover, we have by Jensen's inequality and the fact that $\exp(L^\lambda)$ is a martingale for all $\lambda \in [0,1]$ by Lemmas 5.2.6(i) and 5.7.1

$$e^{\kappa^\lambda(A)} = \mathrm{E}\left(e^{AL_1^\lambda}\right) \leq \mathrm{E}\left(e^{L_1^\lambda}\right)^A = 1,$$

and taking logarithm yields $\kappa^\lambda(A) \leq 0$. In the case $A \in [0,1]^c$ and $A \in D$, we have $A \in D^\lambda$ for all $\lambda \in [0,1]$ by Lemma 5.2.6(iii). Moreover, Representation (5.65) of κ^λ below yields that

$$\lambda\kappa^\lambda(A) = \frac{1}{2}\lambda c^L(A^2 - A) + \frac{1}{\lambda}\left(\int \left(e^{\lambda A x} - 1 - Ae^{\lambda x} + A\right) F^L(\mathrm{d}x)\right).$$

By a Taylor expansion with integral remainder term of the integrand, we obtain

$$\begin{aligned}\lambda\kappa^\lambda(A) &= \frac{1}{2}\lambda c^L(A^2 - A) \\ &\quad + \frac{1}{\lambda}\int \left(\lambda^2 A^2 x^2 \int_0^1 e^{\xi\lambda A x}(1-\xi)\,\mathrm{d}\xi - A\lambda^2 x^2 \int_0^1 e^{\xi\lambda x}(1-\xi)\,\mathrm{d}\xi\right) F^L(\mathrm{d}x).\end{aligned}$$

In the case $A > 1$, this yields the estimate

$$\lambda\kappa^\lambda(A) \leq \frac{1}{2}c^L(A^2 - A) + \frac{1}{2}A^2\int \left(x^2(e^{Ax}\vee 1)\right) F^L(\mathrm{d}x),$$

and in the case $A < 0$

$$\lambda\kappa^\lambda(A) \leq \frac{1}{2}c^L(A^2 - A) + \frac{1}{2}\int \left(A^2x^2(e^{Ax}\vee 1) - Ax^2(e^x \vee 1)\right) F^L(\mathrm{d}x).$$

Combining these estimates completes the proof. □

5.7.2. Time change representation of integrals with respect to Brownian motion

PROOF OF PROPOSITION 5.3.2. The first part of the assertion is an application of Theorem 5.3.1: set $M_t := \int_0^t y_s\,\mathrm{d}W_s$ for $t \in \mathbb{R}_+$. Then, M is a continuous local martingale with $M_0 = 0$, and $\langle M,M\rangle_t = \int_0^t y_s^2\,\mathrm{d}s$. By assumption, $\langle M,M\rangle_\infty = \infty$. Hence Theorem 5.3.1 is applicable, and the process $\widetilde{W}_t := M_{T_t}$ with $T_t := \inf\{s \in \mathbb{R}_+ : \langle M,M\rangle_s > t\}$ is a standard Brownian motion (relative to a different filtration) such that

$$\int_0^t y_s\,\mathrm{d}W_s = \widetilde{W}_{\int_0^t y_s^2\,\mathrm{d}s}, \quad t \in \mathbb{R}_+.$$

It remains to show the assertion regarding the independence of $\widetilde{W}$ and y. To this end, assume from now on that W and y are independent. It is sufficient to show that the characteristic function of the finite-dimensional marginal distributions of $(\widetilde{W}, y)$ factorizes appropriately. More specifically, we will show that for arbitrary $n \in \mathbb{N}$, $t_1, \dots, t_n \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$ we have

$$\mathrm{E}\left(e^{iu^\top(\widetilde{W}_{t_1},\dots,\widetilde{W}_{t_n})+iv^\top(y_{t_1},\dots,y_{t_n})}\right) = \mathrm{E}\left(e^{iu^\top(\widetilde{W}_{t_1},\dots,\widetilde{W}_{t_n})}\right)\mathrm{E}\left(e^{iv^\top(y_{t_1},\dots,y_{t_n})}\right). \tag{5.57}$$

By conditioning, we obtain by construction of $\widetilde{W}$

$$\begin{aligned}
\mathrm{E}\left(e^{iu^\top(\widetilde{W}_{t_1},\dots,\widetilde{W}_{t_n})+iv^\top(y_{t_1},\dots,y_{t_n})}\right) &= \mathrm{E}\left(\mathrm{E}\left(e^{iu^\top(\widetilde{W}_{t_1},\dots,\widetilde{W}_{t_n})+iv^\top(y_{t_1},\dots,y_{t_n})}\,\middle|\,\sigma(y)\right)\right)\\
&= \mathrm{E}\left(e^{iv^\top(y_{t_1},\dots,y_{t_n})}\mathrm{E}\left(e^{iu^\top(\widetilde{W}_{t_1},\dots,\widetilde{W}_{t_n})}\,\middle|\,\sigma(y)\right)\right)\\
&= \mathrm{E}\left(e^{iv^\top(y_{t_1},\dots,y_{t_n})}\mathrm{E}\left(e^{iu^\top\left(\int_0^{T_{t_1}} y_s\,\mathrm{d}W_s,\dots,\int_0^{T_{t_n}} y_s\,\mathrm{d}W_s\right)}\,\middle|\,\sigma(y)\right)\right).
\end{aligned}$$

Recall that for a continuous function $g : \mathbb{R}_+ \to \mathbb{R}$, $m \in \mathbb{N}$, and $s_1, \dots, s_m \in \mathbb{R}_+$ it holds

$$\left(\int_0^{s_1} g(s)\,\mathrm{d}W_s, \dots, \int_0^{s_m} g(s)\,\mathrm{d}W_s\right) \sim N(0,\Sigma),$$

where $\Sigma \in \mathbb{R}^{m\times m}$ with $\Sigma_{ij} = \int_0^{s_i \wedge s_j} g(s)^2\,\mathrm{d}s$ for $1 \le i, j \le m$. Moreover, for $w \in \mathbb{R}^m$

$$\mathrm{E}\left(e^{iw^\top\left(\int_0^{s_1} g(s)\,\mathrm{d}W_s,\dots,\int_0^{s_m} g(s)\,\mathrm{d}W_s\right)}\right) = e^{-\frac{1}{2}w^\top\Sigma w}.$$

Since W is independent of y and hence of $(y, T_{t_1}, \dots, T_{t_n})$, the straightforward generalization of Lemma A.0.4 to the multi-dimensional case yields

$$\mathrm{E}\left(e^{iu^\top\left(\int_0^{T_{t_1}} y_s\,\mathrm{d}W_s,\dots,\int_0^{T_{t_n}} y_s\,\mathrm{d}W_s\right)}\,\middle|\,\sigma(y)\right) = e^{-\frac{1}{2}u^\top Cu},$$

where the (a priori random) matrix C is given by $C_{ij} = \int_0^{T_{t_i}\wedge T_{t_j}} y_s^2\,\mathrm{d}s = \int_0^{T_{t_i \wedge t_j}} y_s^2\,\mathrm{d}s$, $1 \le i, j \le n$. By construction of the stopping times T_t, we have $\int_0^{T_t} y_s^2\,\mathrm{d}s = \langle M, M\rangle_{T_t} = t$ for all $t \in \mathbb{R}_+$. Hence, $C_{ij} = t_i \wedge t_j$ for $1 \le i, j \le n$. Altogether, we obtain

$$\begin{aligned}
\mathrm{E}\left(e^{iu^\top(\widetilde{W}_{t_1},\dots,\widetilde{W}_{t_n})+iv^\top(y_{t_1},\dots,y_{t_n})}\right) &= \mathrm{E}\left(e^{iv^\top(y_{t_1},\dots,y_{t_n})}\right)e^{-\frac{1}{2}u^\top Cu}\\
&= \mathrm{E}\left(e^{iv^\top(y_{t_1},\dots,y_{t_n})}\right)\mathrm{E}\left(e^{iu^\top(\widetilde{W}_{t_1},\dots,\widetilde{W}_{t_n})}\right),
\end{aligned}$$

where the last equality follows from the fact that $\widetilde{W}$ is a standard Brownian motion as well. This shows (5.57), which completes the proof. □

5.7.3. Approximation to the option price

5.7.3.1. Outline for the proof of the main theorem

The proof of our main Theorem 5.4.3 requires several steps and numerous technical lemmas. In this section, we present the main idea and the important steps along the way.

The key idea is to exploit the integral representation

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^z \, p(z) \, \mathrm{d}z$$

of the payoff function from Assumption 5.2.21(1), which allows to represent the option prices c^λ, $\lambda \in [0,1]$, as

$$c^\lambda = \int_{R-i\infty}^{R+i\infty} \varphi_{X_T^\lambda}(z) \, p(z) \, \mathrm{d}z, \tag{5.58}$$

where $\varphi_{X_T^\lambda}$ is the extended characteristic function of $X_T^\lambda = \log(S_T^\lambda)$ (Section 5.7.3.5).

The main steps are then:

1. Represent the extended characteristic function $\varphi_{X_T^\lambda}$ in a fruitful way to separate the processes L, V, and U (Section 5.7.3.4).
2. Show that $\lambda \mapsto \varphi_{X_T^\lambda}(z)$ is smooth, and identify the first two derivatives in $\lambda = 0$ (Section 5.7.3.6).
3. Ensure that differentiation with respect to λ and integration with respect to z in Representation (5.58) can be interchanged (Section 5.7.3.6, using technical results from Sections 5.7.3.2 and 5.7.3.3).
4. Interpret the integrals related to (5.58) over the first two derivatives of $\varphi_{X_T^\lambda}$ in $\lambda = 0$, in particular identify the Black-Scholes cash greeks (Sections 5.7.3.7 and 5.7.3.8).

In Section 5.7.3.9, we give the proof of our approximation to implied volatility, basing on the approximation to the option price.

5.7.3.2. Bounds and derivatives of κ^λ

Recall from Section 5.2.3 that (b^L, c^L, F^L) denotes the Lévy-Khintchine triplet of L relative to the truncation function $h^L : \mathbb{R} \to \mathbb{R}$.

Lemma 5.7.2. *For all $n \in \mathbb{N}_{\geq 2}$, there is measurable $g_n : \mathbb{R} \to \mathbb{R}_+$ such that*

$$\left| x^n e^{\xi z x} \right| \leq g_n(x) \quad \textit{for all } x \in \mathbb{R}, \xi \in [0,1], \textit{ and } z \in \{0, 1, R\} + i\mathbb{R},$$

and

$$\int g_n(x) \, F^L(\mathrm{d}x) < \infty.$$

PROOF. This follows along the same lines as the proof of Lemma 4.5.8, using Assumptions 5.2.4 and 5.2.21(2). Note that F^L is independent of the choice of the truncation function. Moreover, we can consider here all $n \in \mathbb{N}_{\geq 2}$ since we assume that all moments of L_1 exist. □

Lemma 5.7.3. *For the family of cumulant generating functions κ^λ of L^λ, $\lambda \in [0,1]$, here understood as a mapping $\kappa : [0,1] \times (\{0,R\} + i\mathbb{R}) \to \mathbb{C}$, $(\lambda, z) \mapsto \kappa^\lambda(z)$, we have the following: κ is infinitely often partially differentiable with respect to λ, and κ and all partial derivatives are continuous. In particular,*

$$\kappa^0(z) = \frac{1}{2}\mathrm{Var}(L_1)\, z(z-1), \tag{5.59}$$

$$\left.\frac{\partial}{\partial\lambda}\kappa^\lambda(z)\right|_{\lambda=0} = \frac{1}{6}\mathrm{E}\left((L_1 - \mathrm{E}(L_1))^3\right)(z^3 - z), \tag{5.60}$$

$$\left.\frac{\partial^2}{\partial\lambda^2}\kappa^\lambda(z)\right|_{\lambda=0} = \frac{1}{12}\left(\mathrm{E}\left((L_1 - \mathrm{E}(L_1))^4\right) - 3\mathrm{Var}(L_1)^2\right)(z^4 - z). \tag{5.61}$$

Moreover, for all $n \in \mathbb{N}$, there exists $c_n > 0$ such that for all $\lambda \in [0,1]$ and all $z \in \{0,R\} + i\mathbb{R}$

$$\left|\frac{\partial^n}{\partial\lambda^n}\kappa^\lambda(z)\right| \le c_n\left(1 + |z|^{2+n}\right). \tag{5.62}$$

PROOF. The proof is very similar to the one of Proposition 4.5.9. By Lemma 5.2.6(iv), the cumulant generating function κ^λ of L^λ is given in terms of κ^1 by

$$\kappa^\lambda(z) = -\frac{1}{\lambda^2}\kappa^1(\lambda)z + \frac{1}{\lambda^2}\kappa^1(\lambda z), \quad \lambda \in (0,1],\ z \in \{0,R\} + i\mathbb{R} \subset D^\lambda. \tag{5.63}$$

For $\lambda = 0$ we have by (5.10)

$$\kappa^0(z) = \frac{1}{2}\mathrm{Var}(L_1)\, z(z-1), \quad z \in \{0,R\} + i\mathbb{R}. \tag{5.64}$$

By [Sat99, Theorem 25.17], we can express κ^1 in terms of the Lévy-Khintchine triplet (b^L, c^L, F^L) relative to the truncation function $h^L : \mathbb{R} \to \mathbb{R}$ by

$$\kappa^1(z) = b^L z + \frac{1}{2}c^L z^2 + \int \left(e^{zx} - 1 - z h^L(x)\right) F^L(\mathrm{d}x), \quad z \in \{0,R\} + i\mathbb{R}.$$

Inserting this representation into (5.63) yields

$$\kappa^\lambda(z) = \frac{1}{2}c^L z(z-1) + \int \frac{1}{\lambda^2}\left(e^{\lambda zx} - 1 + z - z e^{\lambda x}\right) F^L(\mathrm{d}x), \quad \lambda \in (0,1],\ z \in \{0,R\} + i\mathbb{R}. \tag{5.65}$$

By use of the Taylor expansion with integral remainder term

$$e^{\lambda zx} = 1 + \lambda zx + \frac{1}{2}(\lambda zx)^2 + \frac{1}{2}(\lambda zx)^3 \int_0^1 e^{s\lambda zx}(1-s)^2\,\mathrm{d}s,$$

and analogously for $e^{\lambda x}$, we see that

$$\begin{aligned}\kappa^\lambda(z) &= \frac{1}{2}\left(c^L + \int x^2 F^L(\mathrm{d}x)\right) z(z-1) \\ &\quad + \int\int_0^1 \left(\frac{1}{2}\lambda x^3 (1-s)^2\left(z^3 e^{s\lambda zx} - z e^{s\lambda x}\right)\right)\mathrm{d}s\, F^L(\mathrm{d}x)\end{aligned} \tag{5.66}$$

for $\lambda \in [0,1]$ and $z \in \{0,R\}+i\mathbb{R}$. Note that this representation holds also for $\lambda = 0$ since $\mathrm{Var}(L_1) = c^L + \int x^2 F^L(\mathrm{d}x)$ by [Sat99, Example 25.12]. Obviously, the integrand in (5.66) is infinitely often partially differentiable with respect to λ, and Lemma 5.7.2 allows to find majorants in λ for the partial derivatives of arbitrary order. Corollary C.0.3 then implies that differentiation with respect to λ of arbitrary order and integration can be interchanged, i.e., κ^λ is infinitely often partially differentiable with respect to $\lambda \in [0,1]$. The estimate (5.62) follows from Representation (5.66) and Lemma 5.7.2. Continuity of κ and its partial derivatives follow by dominated convergence, using again Representation (5.66) and Lemma 5.7.2. (5.60) and (5.61) are obtained as in the proof of Proposition 4.5.9, which completes this proof. □

Lemma 5.7.4. *For all $z \in R+i\mathbb{R}$ and all $\lambda \in [0,1]$, we have* $\mathrm{Re}\left(\kappa^\lambda(z)\right) \leq \kappa^\lambda(R)$.

PROOF. For all $z \in R+i\mathbb{R}$ and all $\lambda \in [0,1]$, we have by Jensen's inequality

$$e^{\mathrm{Re}\left(\kappa^\lambda(z)\right)} = \left|e^{\kappa^\lambda(z)}\right| = \left|\mathrm{E}\left(e^{zL_1^\lambda}\right)\right| \leq \mathrm{E}\left(e^{\mathrm{Re}(z)L_1^\lambda}\right) = e^{\kappa^\lambda(R)},$$

and taking logarithm yields the assertion. □

Lemma 5.7.5. *If $c^L > 0$ or if there exists $\gamma > 0$ such that* (5.20) *holds, then there exist $d_1, d_2, \delta > 0$ such that for all $\lambda \in [0,1]$ and all $z \in R+i\mathbb{R}$*

$$\mathrm{Re}\left(\kappa^\lambda(z)\right) \leq d_1 - d_2\left|\mathrm{Im}(z)\right|^\delta. \tag{5.67}$$

PROOF. For $\lambda = 0$, it is obvious from the representation of κ^0 given in (5.59) that the stated exponential decay holds. By (5.65), we have for all $z \in R+i\mathbb{R}$ and all $\lambda \in [0,1]$

$$\kappa^\lambda(z) = \frac{1}{2}c^L z(z-1) + \frac{1}{\lambda^2}\int\left(e^{\lambda zx} - 1 - z(e^{\lambda x}-1)\right)F^L(\mathrm{d}x).$$

There,

$$\mathrm{Re}\left(\frac{1}{2}c^L z(z-1)\right) = \frac{1}{2}c^L\left(R^2 - R - \mathrm{Im}(z)^2\right),$$

and

$$\mathrm{Re}\left(\frac{1}{\lambda^2}\int\left(e^{\lambda zx} - 1 - z(e^{\lambda x}-1)\right)F^L(\mathrm{d}x)\right)$$
$$= \frac{1}{\lambda^2}\int\left(e^{\lambda Rx}(\cos(\lambda x\mathrm{Im}(z)) - 1) + e^{\lambda Rx} - 1 - R(e^{\lambda x}-1)\right)F^L(\mathrm{d}x).$$

It follows from a Taylor expansion of the integrand and Lemma 5.7.2 that

$$d_3 := \sup_{\lambda\in(0,1]}\left|\frac{1}{\lambda^2}\int\left(e^{\lambda Rx} - 1 - R(e^{\lambda x}-1)\right)F^L(\mathrm{d}x)\right| < \infty,$$

and d_3 is obviously independent of $\mathrm{Im}(z)$. Since $(\cos(u)-1) \leq 0$ for all $u \in \mathbb{R}$, we have for all $\lambda \in (0,1]$ and $z \in R+i\mathbb{R}$

$$\mathrm{Re}\left(\kappa^\lambda(z)\right) \leq \frac{1}{2}c^L\left(R^2 - R - \mathrm{Im}(z)^2\right) + d_3,$$

which shows the assertion in the case $c^L > 0$. Otherwise, we have to argue in a more subtle way. To this end, we adapt the proof of [Sat99, Proposition 28.3] to incorporate the additional variable λ. If there exists $\gamma > 0$ such that (5.20) holds, then there are $d_4 > 0$ and $\varepsilon > 0$ such that for all $0 < r < \varepsilon$

$$\int_{-r}^{r} x^2 F^L(\mathrm{d}x) \geq d_4 r^{2-\gamma}.$$

Using that $1 - \cos(u) = 2\sin\left(\frac{u}{2}\right)^2 \geq 2\left(\frac{u}{\pi}\right)^2$ for all $|u| \leq \pi$, we obtain that for all $z \in R + i\mathbb{R}$ with $|\mathrm{Im}\,(z)| \geq 1$ and all $\lambda \in (0,1]$

$$\begin{aligned}\frac{1}{\lambda^2}\int \left(e^{\lambda Rx}(\cos(\lambda x\mathrm{Im}\,(z)) - 1)\right) F^L(\mathrm{d}x) &\leq \frac{1}{\lambda^2}\int_{\left\{|x| \leq \frac{\pi}{\lambda|\mathrm{Im}(z)|}\right\}} -2e^{\lambda Rx}\frac{\lambda^2 x^2 \mathrm{Im}\,(z)^2}{\pi^2} F^L(\mathrm{d}x)\\ &\leq -2\frac{\mathrm{Im}\,(z)^2}{\pi^2} d_5 \int_{\left\{|x| \leq \frac{\pi}{|\mathrm{Im}(z)|}\right\}} x^2 F^L(\mathrm{d}x),\end{aligned}$$

where

$$\inf_{\left\{|x| \leq \frac{\pi}{\lambda|\mathrm{Im}(z)|}\right\}} e^{\lambda Rx} = e^{-|R|\frac{\pi}{|\mathrm{Im}(z)|}} \geq e^{-|R|\pi} =: d_5$$

for all $z \in R + i\mathbb{R}$ with $|\mathrm{Im}\,(z)| \geq 1$ and all $\lambda \in (0,1]$. Hence, for all $z \in R + i\mathbb{R}$ with $|\mathrm{Im}\,(z)| \geq \left(1 \vee \frac{\pi}{\varepsilon}\right)$ and all $\lambda \in (0,1]$, we obtain

$$\begin{aligned}\frac{1}{\lambda^2}\int \left(e^{\lambda Rx}(\cos(\lambda x\mathrm{Im}\,(z)) - 1)\right) F^L(\mathrm{d}x) &\leq -\frac{2}{\pi^2} d_4 d_5 \left|\frac{\pi}{\mathrm{Im}\,(z)}\right|^{2-\gamma} \mathrm{Im}\,(z)^2\\ &= -\frac{2}{\pi^\gamma} d_4 d_5 \,|\mathrm{Im}\,(z)|^\gamma.\end{aligned}$$

Collecting all constants and choosing d_1 such that the claimed estimate holds also for $z \in R + i\mathbb{R}$ with $|\mathrm{Im}\,(z)| < \left(1 \vee \frac{\pi}{\varepsilon}\right)$ – which is possible by the continuity of $(\lambda, z) \mapsto \kappa^\lambda(z)$ by Lemma 5.7.3 – completes the proof. □

5.7.3.3. Bounds and derivatives of U_T^λ

Proposition 5.7.6. *Under Assumption 5.2.21*(4)*, there exists a random variable Z such that* $\exp\left(RU_T^\lambda\right) \leq Z$ *for all* $\lambda \in [0,1]$ *and such that* $\mathrm{E}\left(Z^2\right) < \infty$.

Proof. In the case $R \geq 0$ we have for all $\lambda \in [0,1]$

$$\exp(RU_T^\lambda) = \exp(R\lambda M_T - RK(\lambda M)_T) \leq \exp(R\lambda M_T) \leq 1 + \exp(RM_T)$$

since $K(\lambda M)_T \geq 0$ by Lemma A.0.5. Observing that $(1 + \exp(RM_T))^2 \leq 3(1 + \exp(2RM_T))$, the assertion follows in the case $R \geq 0$ since the latter random variable has finite expectation by Assumption 5.2.21(4). In the case $R < 0$ we first note that for all $\lambda \in [0,1]$

$$\exp(R\lambda M_T - RK(\lambda M)_T) = \exp(R\lambda(M_T - K(M)_T))\exp(R(\lambda K(M)_T - K(\lambda M)_T)).$$

In order to find a majorant for the right-hand side, we are interested in an estimate from below of $\lambda K(M)_T - K(\lambda M)_T$ since $R < 0$. From the representation of $K(\lambda M)$ by Lemma 5.2.10, we see that for all $\lambda \in [0,1]$

$$\begin{aligned}\lambda K(M)_T - K(\lambda M)_T &= \int_0^T \frac{1}{2}(\lambda - \lambda^2) c_s^M \,\mathrm{d}s \\ &\quad + \int_0^T \int \Big(\lambda(e^x - 1 - h^M(x)) - (e^{\lambda x} - 1 - \lambda h^M(x))\Big) F_s^M(\mathrm{d}x)\,\mathrm{d}s \\ &= \int_0^T \left(\frac{1}{2}(\lambda - \lambda^2) c_s^M + \int \Big(\lambda e^x - \lambda - e^{\lambda x} + 1\Big) F_s^M(\mathrm{d}x)\right) \mathrm{d}s.\end{aligned}$$

By a straightforward analysis using fundamental calculus, we see that the function $(\lambda, x) \mapsto \Big(\lambda e^x - \lambda - e^{\lambda x} + 1\Big)$ is non-negative on $[0,1] \times \mathbb{R}$. Hence, $\lambda K(M)_T - K(\lambda M)_T \geq 0$ since $c^M \geq 0$. This yields for all $\lambda \in [0,1]$

$$\exp(RU_T^\lambda) \leq \exp(R\lambda U_T) \leq 1 + \exp(RU_T)$$

Using again that $(1 + \exp(RU_T))^2 \leq 3(1 + \exp(2RU_T))$ completes the proof since the latter random variable has finite expectation by Assumption 5.2.21(4). □

Lemma 5.7.7. *We have the following:*

(i) For almost all $\omega \in \Omega$, the mapping $\lambda \mapsto U_T^\lambda(\omega)$ is in $C^\infty([0,1],\mathbb{R})$.

(ii) In particular, we have

$$\begin{aligned} U_T^0 &= 0, \\ \frac{\partial}{\partial\lambda} U_T^\lambda \bigg|_{\lambda=0} &= M_T, \\ \frac{\partial^2}{\partial\lambda^2} U_T^\lambda \bigg|_{\lambda=0} &= -\langle M, M\rangle_T. \end{aligned}$$

(iii) For all $n \in \mathbb{N}$, there exists a random variable B_n having arbitrary moments and such that for all $\lambda \in [0,1]$

$$\left| \frac{\partial^n}{\partial\lambda^n} U_T^\lambda \right| \leq B_n.$$

Proof. We show all three assertions in a common proof. By definition of U^λ in (5.12) and the representation of $K(\lambda M)$ in (5.11), we have

$$\begin{aligned} U_T^\lambda &= \lambda M_T - \left(\lambda \int_0^T b_s^M \,\mathrm{d}s + \frac{1}{2}\lambda^2 \int_0^T c_s^M \,\mathrm{d}s + \int_0^T \int \Big(e^{\lambda x} - 1 - \lambda h^M(x)\Big) F_s^M(\mathrm{d}x)\,\mathrm{d}s\right) \quad (5.68) \\ &= \lambda M_T - \left(\lambda \int_0^T b_s^M \,\mathrm{d}s + \frac{1}{2}\lambda^2 \int_0^T c_s^M \,\mathrm{d}s\right) \\ &\quad - \int_0^T \int \left(\lambda(x - h^M(x)) + \lambda^2 x^2 \int_0^1 e^{\xi\lambda x}(1-\xi)\,\mathrm{d}\xi\right) F_s^M(\mathrm{d}x)\,\mathrm{d}s, \end{aligned}$$

where the last step follows from a Taylor expansion with integral remainder term of the integrand of the third integral in (5.68). Since M is a local martingale, we have $\int_{\{|x|\geq 1\}} |x|\, F^M(\mathrm{d}x) < \infty$ and $b+\int (x-h^M(x))\, F^M(\mathrm{d}x) = 0$ $(P\otimes \mathrm{d}t)$-almost everywhere by Proposition E.0.8(1). Hence, we have

$$U_T^\lambda = \lambda M_T - \left(\frac{1}{2}\lambda^2 \int_0^T c_s^M \,\mathrm{d}s + \int_0^T \int \left(\lambda^2 x^2 \int_0^1 e^{\xi\lambda x}(1-\xi)\,\mathrm{d}\xi\right) F_s^M(\mathrm{d}x)\,\mathrm{d}s\right). \tag{5.69}$$

Obviously, $U_T^0 = 0$. For all $x\in\mathbb{R}$, $\lambda\in[0,1]$, $\xi\in[0,1]$, we have

$$\begin{aligned}
\left|\lambda^2 x^2 \int_0^1 e^{\xi\lambda x}(1-\xi)\,\mathrm{d}\xi\right| &\leq x^2\,(e^x\vee 1)\,,\\
\frac{\partial}{\partial\lambda}\left(\lambda^2 x^2 \int_0^1 e^{\xi\lambda x}(1-\xi)\,\mathrm{d}\xi\right) &= 2\lambda x^2 \int_0^1 e^{\xi\lambda x}(1-\xi)\,\mathrm{d}\xi + \lambda^2 x^3 \int_0^1 e^{\xi\lambda x}(1-\xi)\xi\,\mathrm{d}\xi,\\
\left|\frac{\partial}{\partial\lambda}\left(\lambda^2 x^2 \int_0^1 e^{\xi\lambda x}(1-\xi)\,\mathrm{d}\xi\right)\right| &\leq \left(2x^2+|x|^3\right)(e^x\vee 1)\,.
\end{aligned}\tag{5.70}$$

For the next step, we will use that

$$\langle M,M\rangle_T = \int_0^T c_s^M\,\mathrm{d}s + \int_0^T\int x^2\, F_s^M(\mathrm{d}x)\,\mathrm{d}s \geq \int_0^T c_s^M\,\mathrm{d}s$$

by Proposition E.0.9. For all $\lambda\in[0,1]$, we obtain from (5.69)

$$\left|U_T^\lambda\right| \leq |M_T| + \frac{1}{2}\langle M,M\rangle_T + \int_0^T\int x^2\,(e^x\vee 1)\, F_s^M(\mathrm{d}x)\,\mathrm{d}s =: B_0. \tag{5.71}$$

By Assumptions 5.2.15, 5.2.25 and 5.2.26, all random variables in the sum in (5.71) have all moments, and hence so does B_0. As well by Assumption 5.2.26, we have

$$\int_0^T\int \left(x^2+|x|^3\right)(e^x\vee 1)\, F_s^M(\mathrm{d}x)\,\mathrm{d}s < \infty \quad \text{almost surely,}$$

and hence we can interchange differentiation with respect to λ and integration in the last integral in (5.69) by Corollary C.0.3. Hence, $\lambda\mapsto U_T^\lambda$ is in $C^1([0,1],\mathbb{R})$ almost surely, and

$$\frac{\partial}{\partial\lambda}U_T^\lambda = M_T - \lambda\int_0^T c_s^M\,\mathrm{d}s - \int_0^T\int\int_0^1 e^{\xi\lambda x}(1-\xi)\left(2\lambda x^2+\lambda^2x^3\xi\right)\mathrm{d}\xi\, F_s^M(\mathrm{d}x)\,\mathrm{d}s.$$

In particular,

$$\left.\frac{\partial}{\partial\lambda}U_T^\lambda\right|_{\lambda=0} = M_T.$$

Moreover, by the estimate in (5.70) we have for all $\lambda\in[0,1]$

$$\left|\frac{\partial}{\partial\lambda}U_T^\lambda\right| \leq |M_T| + \langle M,M\rangle_T + \int_0^T\int\left(2x^2+|x|^3\right)(e^x\vee 1)\, F_s^M(\mathrm{d}x)\,\mathrm{d}s =: B_1.$$

Again, all random variables in the sum on the right-hand side have all moments by Assumptions 5.2.15, 5.2.25 and 5.2.26, and so does B_1. For the higher derivatives, it is more convenient to work with the representation of U_T^λ given in (5.68). For all $\lambda \in [0,1]$, $x \in \mathbb{R}$, $n \in \mathbb{N}_{\geq 2}$, we have for the integrand of the third integral there

$$\begin{aligned} \frac{\partial^n}{\partial \lambda^n}\left(e^{\lambda x} - 1 - \lambda h^M(x)\right) &= x^n e^{\lambda x}, \\ \left|\frac{\partial^n}{\partial \lambda^n}\left(e^{\lambda x} - 1 - \lambda h^M(x)\right)\right| &\leq |x|^n \left(e^x \vee 1\right). \end{aligned} \tag{5.72}$$

Using Assumption 5.2.26 and Corollary C.0.3 to interchange integration and differentiation with respect to λ, we obtain for all $\lambda \in [0,1]$ and $n \in \mathbb{N}_{\geq 3}$

$$\begin{aligned} \frac{\partial^2}{\partial \lambda^2} U_T^\lambda &= -\left(\int_0^T c_s^M \,\mathrm{d}s + \int_0^T \int x^2 e^{\lambda x} F_s^M(\mathrm{d}x)\,\mathrm{d}s\right), \\ \frac{\partial^n}{\partial \lambda^n} U_T^\lambda &= -\int_0^T \int x^n e^{\lambda x} F_s^M(\mathrm{d}x)\,\mathrm{d}s. \end{aligned}$$

In particular,

$$\left.\frac{\partial^2}{\partial \lambda^2} U_T^\lambda\right|_{\lambda=0} = -\left(\int_0^T c_s^M \,\mathrm{d}s + \int_0^T \int x^2 F_s^M(\mathrm{d}x)\,\mathrm{d}s\right) = -\langle M, M\rangle_T.$$

Analogously to the arguments for the first derivative, Estimate (5.72) and Assumptions 5.2.25 and 5.2.26 yield that for all $n \in \mathbb{N}_{\geq 2}$ there is a random variable B_n with all moments such that for all $\lambda \in [0,1]$

$$\left|\frac{\partial^n}{\partial \lambda^n} U_T^\lambda\right| \leq B_n,$$

which completes the proof. □

5.7.3.4. Representation of the extended characteristic function of X_T^λ

Notation 5.7.8. For an $\mathbb{R}^d$-valued random variable Y, we denote by φ_Y the extended characteristic function of Y, i.e.,

$$\varphi_Y(z) = \mathrm{E}\left(e^{z^\top Y}\right)$$

for all $z \in \mathbb{C}^d$ such that $\mathrm{E}\left(e^{\mathrm{Re}(z)^\top Y}\right) < \infty$.

Lemma 5.7.9. *For all $z \in R + i\mathbb{R}$ and all $\lambda \in [0,1]$, we have $\mathrm{E}\left(e^{\mathrm{Re}(z) X_T^\lambda}\right) < \infty$, and the extended characteristic function of X_T^λ has the representations*

$$\begin{aligned} \varphi_{X_T^\lambda}(z) &= S_0^z \mathrm{E}\left(\exp\left(\kappa^\lambda(z) V_T^\lambda\right)\exp\left(z U_T^\lambda\right)\right) & (5.73)\\ &= S_0^z \exp\left(\kappa^\lambda(z)\left((1-\lambda)\mathrm{E}(V_T) + (1-\lambda^2)\frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}\right)\right)\varphi_{(V_T, U_T^\lambda)}\left(\lambda\kappa^\lambda(z), z\right). & (5.74) \end{aligned}$$

PROOF. Let $z \in R + i\mathbb{R}$ and $\lambda \in [0,1]$ be fixed for the rest of the proof. By definition of X^λ in (5.14), we have

$$\mathrm{E}\left(\left|\exp\left(zX_T^\lambda\right)\right|\right) = \mathrm{E}\left(\exp\left(RX_T^\lambda\right)\right) = S_0^R \mathrm{E}\left(\exp\left(R\left(L_{V_T^\lambda}^\lambda + U_T^\lambda\right)\right)\right).$$

Using the tower property of the conditional expectation and the independence of L^λ and $\left(V^\lambda, U^\lambda\right)$, we can rewrite this term to incorporate the cumulant generating function κ^λ of L^λ via Lemma A.0.4. This yields

$$\begin{aligned}\mathrm{E}\left(\exp\left(RX_T^\lambda\right)\right) &= S_0^R \mathrm{E}\left(\mathrm{E}\left(\exp\left(R\left(L_{V_T^\lambda}^\lambda + U_T^\lambda\right)\right)\Big|\left(V_T^\lambda, U_T^\lambda\right)\right)\right)\\ &= S_0^R \mathrm{E}\left(\exp\left(\kappa^\lambda(R)V_T^\lambda + RU_T^\lambda\right)\right).\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$\mathrm{E}\left(\exp\left(\kappa^\lambda(R)V_T^\lambda + RU_T^\lambda\right)\right) \le \mathrm{E}\left(\exp\left(2\kappa^\lambda(R)V_T^\lambda\right)\right)^{\frac{1}{2}} \mathrm{E}\left(\exp\left(2RU_T^\lambda\right)\right)^{\frac{1}{2}}. \tag{5.75}$$

The second factor on the right-hand side is finite by Proposition 5.7.6. By the definition of V^λ in (5.13), we see that

$$2\kappa^\lambda(R)V_T^\lambda = 2\kappa^\lambda(R)\left(\lambda V_T + (1-\lambda)\mathrm{E}(V_T) + (1-\lambda^2)\frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}\right).$$

Hence, Assumption 5.2.21(3) ensures that also the first factor on the right-hand side of (5.75) is finite, which shows that $\mathrm{E}\left(e^{\mathrm{Re}(z)X_T^\lambda}\right) < \infty$. The same calculation as above with R replaced by z yields for all $z \in R + i\mathbb{R}$

$$\varphi_{X_T^\lambda}(z) = S_0^z \mathrm{E}\left(\exp\left(\kappa^\lambda(z)V_T^\lambda + zU_T^\lambda\right)\right) = S_0^z \varphi_{\left(V_T^\lambda, U_T^\lambda\right)}\left(\kappa^\lambda(z), z\right).$$

Inserting the definition of V^λ from (5.13) yields the second equation in the assertion and completes the proof. □

5.7.3.5. Representation of the option price

In order to represent the option price c^λ in a fruitful way, we exploit an integral representation of the payoff function f given by the following

Proposition 5.7.10. *There exists $p : (R + i\mathbb{R}) \to \mathbb{C}$ with $x \mapsto |p(R+ix)|$ being integrable such that the payoff function f from Section 5.2.1 admits the representation*

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^z p(z)\,\mathrm{d}z, \quad s \in \mathbb{R}_+. \tag{5.76}$$

In the case that Assumption 5.2.21(1b) is in force, we additionally have that $x \mapsto |R+ix|^n |p(R+ix)|$ is integrable for all $n \in \mathbb{N}$.

PROOF. The assertion is either given directly by Assumption 5.2.21(1a), or it is implied by the alternative Assumption 5.2.21(1b) and Lemma 3.3.1. □

Proposition 5.7.11. *For all $\lambda \in [0,1]$, we have $\mathrm{E}\left(\left|f(S_T^\lambda)\right|\right) < \infty$, and the option price $c^\lambda = \mathrm{E}\left(f(S_T^\lambda)\right)$ relative to S^λ from* (5.22) *has the representation*

$$c^\lambda = \int_{R-i\infty}^{R+i\infty} \varphi_{X_T^\lambda}(z)\, p(z)\, \mathrm{d}z \tag{5.77}$$

with $p : (R+i\mathbb{R}) \to \mathbb{C}$ from Proposition 5.7.10.

PROOF. This follows directly from Lemma 5.7.9, Proposition 5.7.11, and Theorem 3.1.1. □

5.7.3.6. Derivatives and bounds of the extended characteristic function of X_T^λ

Lemma 5.7.12. *We have $M \in \mathscr{H}^2_{\mathrm{loc}}$, and for all $t \in \mathbb{R}_+$ it holds*

$$\mathrm{E}\left([M,M]_t\right) = \mathrm{E}\left(\langle M,M\rangle_t\right) = \mathrm{Var}\left(M_t\right).$$

PROOF. By Assumptions 5.2.8 and 5.2.15, M is a martingale with finite second moments. Hence M^2 is a submartingale by Jensen's inequality, and in particular, $t \mapsto \mathrm{E}\left(M_t^2\right)$ is a real-valued increasing function. Note that since $M_0 = 0$, we have $\mathrm{Var}\left(M_t\right) = \mathrm{E}\left(M_t^2\right)$ for all $t \in \mathbb{R}_+$. Because $t \mapsto \mathrm{E}\left(M_t^2\right)$ is increasing, we have for all $n \in \mathbb{N}$ that $\sup_{t\in\mathbb{R}_+} \mathrm{E}\left(\left(M^n\right)_t^2\right) \le \mathrm{E}\left(M_n^2\right) < \infty$, which shows that $M \in \mathscr{H}^2_{\mathrm{loc}}$. By the same argument, we see that the stopped process $M^t \in \mathscr{H}$ for all $t \in \mathbb{R}_+$. Let now $t \in \mathbb{R}_+$ be arbitrary but fixed. Then, $[M,M]$ and $[M^t,M^t]$ are special by [JS03, Proposition I.4.50(b)], and by the same statement

$$\mathrm{E}\left(M_t^2\right) = \mathrm{E}\left(\left(M^t\right)_t^2\right) = \mathrm{E}\left(\left[M^t,M^t\right]_t\right) = \mathrm{E}\left([M,M]_t\right), \tag{5.78}$$

where the last equality follows from Lemma A.0.7. By definition, $[M,M] - \langle M,M\rangle \in \mathscr{M}_{\mathrm{loc}}$. Denoting the corresponding localizing sequence by $\left(\tau_n\right)_{n\in\mathbb{N}}$, we have for all $n \in \mathbb{N}$ that $[M,M]^{\tau_n} - \langle M,M\rangle^{\tau_n} \in \mathscr{M}$, and hence $\mathrm{E}\left([M,M]_{t\wedge\tau_n}\right) = \mathrm{E}\left(\langle M,M\rangle_{t\wedge\tau_n}\right)$ for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. By monotone convergence, we obtain

$$\mathrm{E}\left([M,M]_t\right) = \mathrm{E}\left(\langle M,M\rangle_t\right), \quad t \in \mathbb{R}_+,$$

which yields the result recalling (5.78). □

Lemma 5.7.13. *(i) For all $z \in R+i\mathbb{R}$, the mapping $\lambda \mapsto \varphi_{X_T^\lambda}(z)$ is in $C^\infty([0,1],\mathbb{C})$.*

(ii) We have

$$
\begin{aligned}
\varphi_{X_T^0}(z) &= S_0^z \exp\left(\frac{1}{2}\left(\mathrm{Var}(L_1)\,\mathrm{E}(V_T)+\mathrm{Var}(M_T)\right)z(z-1)\right),\\
\frac{\partial}{\partial\lambda}\varphi_{X_T^\lambda}(z)\bigg|_{\lambda=0} &= \varphi_{X_T^0}(z)\frac{1}{6}\left(z^3-z\right)\mathrm{E}\left((L_1-\mathrm{E}(L_1))^3\right)\left(\mathrm{E}(V_T)+\frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}\right),\\
\frac{\partial^2}{\partial\lambda^2}\varphi_{X_T^\lambda}(z)\bigg|_{\lambda=0} &= \varphi_{X_T^0}(z)\left\{\left(\frac{1}{6}\left(z^3-z\right)\mathrm{E}\left((L_1-\mathrm{E}(L_1))^3\right)\left(\mathrm{E}(V_T)+\frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}\right)\right)^2\right.\\
&\quad+\frac{1}{4}\mathrm{Var}(L_1)^2\,\mathrm{Var}(V_T)\,z^2(z-1)^2+z^2(z-1)\mathrm{Var}(L_1)\,\mathrm{Cov}(V_T,M_T)\\
&\quad\left.+\frac{1}{12}\left(z^4-z\right)\left(\mathrm{E}\left((L_1-\mathrm{E}(L_1))^4\right)-3\mathrm{Var}(L_1)^2\right)\left(\mathrm{E}(V_T)+\frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}\right)\right\}.
\end{aligned}
$$

(iii) For all $n\in\mathbb{N}$, there exists $M_n:(R+i\mathbb{R})\to\mathbb{R}_+$ such that for all $\lambda\in[0,1]$ and all $z\in R+i\mathbb{R}$

$$
\left|\frac{\partial^n}{\partial\lambda^n}\varphi_{X_T^\lambda}(z)\right|\le M_n(z)
$$

and such that

$$
\int_{-\infty}^{\infty} M_n(R+ix)\,|p(R+ix)|\,\mathrm{d}x<\infty
$$

with $p:(R+i\mathbb{R})\to\mathbb{C}$ from Proposition 5.7.10.

PROOF. Let us begin with introducing some notation. For $\lambda\in[0,1]$ and $z\in R+i\mathbb{R}$, we set

$$
\psi(\lambda,z):=z\log(S_0)+\psi^0(\lambda,z)+\psi^1(\lambda,z)+\psi^2(\lambda,z)
$$

with

$$
\begin{aligned}
\psi^0(\lambda,z)&:=\kappa^\lambda(z)\left((1-\lambda)\mathrm{E}(V_T)+(1-\lambda^2)\frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}\right),\\
\psi^1(\lambda,z)&:=\lambda\,\kappa^\lambda(z)V_T,\\
\psi^2(\lambda,z)&:=zU_T^\lambda.
\end{aligned}
$$

I.e., by Lemma 5.7.9,

$$
\varphi_{X_T^\lambda}(z)=\mathrm{E}\left(e^{\psi(\lambda,z)}\right).
$$

It is our first aim to show that $\lambda\mapsto\varphi_{X_T^\lambda}(z)$ is infinitely often differentiable. To this end, in a first step we will show that the random variable in the above expected value is point-wise infinitely often differentiable with respect to λ for all $z\in R+i\mathbb{R}$ and $\omega\in\Omega$. In a second step, we show that all partial derivatives allow, for fixed $z\in R+i\mathbb{R}$, for an integrable majorant with respect to λ. Corollary C.0.3 will then yield Assertion (i).

For all $z\in R+i\mathbb{R}$ and (suppressed) $\omega\in\Omega$, the mapping $\lambda\mapsto\psi(\lambda,z)$ is in $C^\infty([0,1],\mathbb{C})$ since this is the case also for $\lambda\mapsto\kappa^\lambda(z)$ and $\lambda\mapsto U_T^\lambda$ by Lemmas 5.7.3 and 5.7.7. Hence, also

$\lambda \mapsto \exp(\psi(\lambda,z))$ is in $C^\infty([0,1],\mathbb{C})$. For all $z \in R+i\mathbb{R}$, $\lambda \in [0,1]$, $n \in \mathbb{N}$, the chain rule of fundamental calculus implies that the corresponding derivatives are given by

$$\frac{\partial^n}{\partial \lambda^n} e^{\psi(\lambda,z)} = e^{\psi(\lambda,z)} \sum_{|\alpha|=n} c_\alpha \prod_{m=1}^{n} \frac{\partial^{\alpha_m}}{\partial \lambda^{\alpha_m}} \psi(\lambda,z), \tag{5.79}$$

where $\alpha = (\alpha_1,\ldots,\alpha_n)$ denotes a multi-index of order n, and $c_\alpha \geq 0$ are appropriate constants that are independent of λ and z.

By Lemma 5.7.4, we have $\operatorname{Re}\left(\kappa^\lambda(z)\right) \leq \kappa^\lambda(R)$ for all $z \in R+i\mathbb{R}$ and all $\lambda \in [0,1]$. Moreover, by means of Proposition 5.7.6, there exists a random variable Z such that $\exp\left(RU_T^\lambda\right) \leq Z$ for all $\lambda \in [0,1]$ and such that $\mathrm{E}\left(Z^2\right) < \infty$. (The last fact will be used later on.) Altogether, this yields for all $\lambda \in [0,1]$ and all $z \in R+i\mathbb{R}$

$$\left|e^{\psi^0(\lambda,z)}\right| \leq \exp\left(\max_{\lambda\in[0,1]} \kappa^\lambda(R)\left(\mathrm{E}(V_T) + \frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)}\right)\right) =: y, \tag{5.80}$$

$$\left|e^{\psi^1(\lambda,z)}\right| \leq \exp\left(\max_{\lambda\in[0,1]} \lambda\kappa^\lambda(R)V_T\right) =: Y, \tag{5.81}$$

$$\left|e^{\psi^2(\lambda,z)}\right| \leq Z.$$

Note that the maxima over $\lambda \mapsto \kappa^\lambda(R)$ are well-defined since this mapping is continuous by Lemma 5.7.3, and note that these estimates are already independent of z. Let us now construct a majorant for the remaining expression in (5.79). For all $n \in \mathbb{N}$, $\frac{\partial^n}{\partial\lambda^n}\psi^0_{X_T^\lambda}(z)$ is not stochastic. Moreover, Lemma 5.7.3 yields that for all $n \in \mathbb{N}$ there exist constants c_n^0 and c_n^1 such that for all $\lambda \in [0,1]$ and all $z \in R+i\mathbb{R}$

$$\left|\frac{\partial^n}{\partial\lambda^n}\psi^0(\lambda,z)\right| \leq c_n^0\left(1+|z|^{2+n}\right) =: h_n(z)$$

and

$$\left|\frac{\partial^n}{\partial\lambda^n}\psi^1(\lambda,z)\right| \leq c_n^1\left(1+|z|^{2+n}\right)V_T =: H_n(z).$$

By Lemma 5.7.7, for all $n \in \mathbb{N}$ there exists a random variable B_n with arbitrary moments and such that for all $\lambda \in [0,1]$ we have $\left|\frac{\partial^n}{\partial\lambda^n}U_T^\lambda\right| \leq B_n$. Hence, for all $z \in R+i\mathbb{R}$ we have constructed a majorant in λ for $\frac{\partial^n}{\partial\lambda^n}e^{\psi(\lambda,z)}$:

$$\left|\frac{\partial^n}{\partial\lambda^n}e^{\psi(\lambda,z)}\right| \leq S_0^R yYZ \sum_{|\alpha|=n} c_\alpha \prod_{m=1}^{n}\left(h_{\alpha_m}(z) + H_{\alpha_m}(z) + |z|B_{\alpha_m}\right).$$

In order to check that this majorant has finite expectation, we first observe that

$$\begin{aligned}\sum_{|\alpha|=n} c_\alpha \prod_{m=1}^{n}\left(h_{\alpha_m}(z) + H_{\alpha_m}(z) + |z|B_{\alpha_m}\right) &\leq \sum_{|\alpha|=n} c_\alpha \prod_{m=1}^{n}\left(1+|z|^{2+\alpha_m}\right)\left(c^0_{\alpha_m} + c^1_{\alpha_m}V_T + B_{\alpha_m}\right)\\ &\leq \left(1+|z|^{2+n}\right)^n\left(\widehat{c}_n^0 + \widehat{c}_n^1 V_T + \widehat{B}_n\right)^n \sum_{|\alpha|=n} c_\alpha,\end{aligned} \tag{5.82}$$

where we set for $n \in \mathbb{N}$

$$\widehat{c}_n^0 := \max_{i=0,\dots,n} c_i^0, \quad \widehat{c}_n^1 := \max_{i=0,\dots,n} c_i^1, \quad \widehat{B}_n := \sum_{i=0}^n B_i.$$

We examine the expectation over the relevant expressions. Observe first that for all $n \in \mathbb{N}$ by iterated application of the Cauchy-Schwartz inequality,

$$\begin{aligned} \mathrm{E}\left(YZ\left(\widehat{c}_n^0+\widehat{c}_n^1+\widehat{B}_n\right)^n\right) \leq \mathrm{E}\left(Z^2\right)^{\frac{1}{2}} \cdot \mathrm{E}\left(\exp\left(4 \max_{\lambda\in[0,1]} \lambda\kappa^\lambda(R)V_T\right)\right)^{\frac{1}{4}} \\ \cdot \mathrm{E}\left(\left(\widehat{c}_n^0+\widehat{c}_n^1 V_T+\widehat{B}_n\right)^{4n}\right)^{\frac{1}{4}}. \end{aligned} \tag{5.83}$$

Recall from above that Z was the random variable given by Proposition 5.7.6 such that $\exp(RU_T^\lambda) \leq Z$ for all $\lambda \in [0,1]$ and such that $\mathrm{E}\left(Z^2\right) < \infty$. Hence, the first expectation in (5.83) is finite. The second one is finite by Assumption 5.2.21(3). For the third expectation, we have

$$\mathrm{E}\left(\left(\widehat{c}_n^0+\widehat{c}_n^1 V_T+\widehat{B}_n\right)^{4n}\right) \leq 3^{4n}\left(\left(\widehat{c}_n^0\right)^{4n}+\left(\widehat{c}_n^1\right)^{4n}\mathrm{E}\left((V_T)^{4n}\right)+\mathrm{E}\left(\left(\widehat{B}_n\right)^{4n}\right)\right). \tag{5.84}$$

The first expectation on the right-hand side is finite by Assumption 5.2.15. For the second one, we have by Jensen's inequality for sums

$$\mathrm{E}\left(\left(\widehat{B}_n\right)^{4n}\right) \leq n^{4n}\sum_{i=1}^n \mathrm{E}\left((B_i)^{4n}\right), \tag{5.85}$$

and all the expectations occurring there are finite by Lemma 5.7.7.

Now, we can apply Corollary C.0.3 to conclude that for all $z \in R+i\mathbb{R}$, all $\lambda \in [0,1]$ and all $n \in \mathbb{N}$ the derivative $\frac{\partial^n}{\partial\lambda^n}\varphi_{X_T^\lambda}(z)$ exists, and

$$\frac{\partial^n}{\partial\lambda^n}\varphi_{X_T^\lambda}(z) = \mathrm{E}\left(\frac{\partial^n}{\partial\lambda^n}e^{\Psi(\lambda,z)}\right),$$

which yields Assertion (i).

The expressions for $\varphi_{X_T^\lambda}(z)$ and the first two derivatives at $\lambda = 0$ given in Assertion (ii) are obtained in straightforward manner by evaluating $e^{\Psi(\lambda,z)}$ and its first two derivatives at $\lambda = 0$ and then taking expectation. The derivatives of $\kappa^\lambda(z)$ are provided by Lemma 5.7.3, which brings the moments of L_1 into play. The derivatives of U_T^λ are given by Lemma 5.7.7. One thing to mention here is that $\mathrm{E}\left(\langle M,M\rangle_T\right) = \mathrm{Var}\left(M_T\right)$ by means of Lemma 5.7.12.

We now turn to the proof of Assertion (iii). Let us first consider the case that Assumption 5.2.21(1b) is in force. Then by Proposition 5.7.10, we have for all $n \in \mathbb{N}$

$$\int_{-\infty}^{\infty} |R+ix|^n\,|p(R+ix)|\,\mathrm{d}x < \infty. \tag{5.86}$$

We can conclude from our previous estimates in (5.82), (5.83), (5.84) and (5.85) that for all $n \in \mathbb{N}$ there exists $a_n > 0$ such that for all $\lambda \in [0,1]$ and all $z \in R + i\mathbb{R}$

$$\left| \frac{\partial^n}{\partial \lambda^n} \varphi_{X_T^\lambda}(z) \right| \leq a_n \left(1 + |z|^{2+n}\right)^n.$$

Setting $M_n(z) = a_n \left(1 + |z|^{2+n}\right)^n$, Assertion (iii) follows from (5.86). In the other case that Assumption 5.2.21(1a) is in force, we have to argue in a more subtle way. To this end, we observe that for all $z \in R + i\mathbb{R}$ and all $\lambda \in [0,1]$

$$\begin{aligned} \left| \frac{\partial^n}{\partial \lambda^n} \varphi_{X_T^\lambda}(z) \right| &\leq \mathrm{E}\left(\left| \frac{\partial^n}{\partial \lambda^n} e^{\psi(\lambda,z)} \right| \right) \\ &\leq \left(\sum_{|\alpha|=n} c_\alpha \right) \left(1 + |z|^{2+n}\right)^n \mathrm{E}\left(\left| e^{\psi(\lambda,z)} \right| \left(\widehat{c}_n^0 + \widehat{c}_n^1 V_T + \widehat{B}_n \right)^n \right). \end{aligned} \tag{5.87}$$

By the Cauchy-Schwartz inequality, we conclude as in (5.83)

$$\begin{aligned} \mathrm{E}\left(\left| e^{\psi(\lambda,z)} \right| \left(\widehat{c}_n^0 + \widehat{c}_n^1 V_T + \widehat{B}_n \right)^n \right) &\leq S_0^R \mathrm{E}\left(Z^2\right)^{\frac{1}{2}} \\ \mathrm{E}\left(\left(\widehat{c}_n^0 + \widehat{c}_n^1 V_T + \widehat{B}_n \right)^{4n} \right)^{\frac{1}{4}} & e^{\mathrm{Re}\left(\psi^0(\lambda,z)\right)} \mathrm{E}\left(e^{4\mathrm{Re}\left(\psi^1(\lambda,z)\right)} \right)^{\frac{1}{4}}. \end{aligned} \tag{5.88}$$

We have already seen that the first two expectations on the right-hand side are finite (and obviously independent of λ and z). We will now show that the two remaining expressions are an exponentially decaying function in $|\mathrm{Im}(z)|$ uniformly in $\lambda \in [0,1]$. To this end, recall that by Assumption 5.2.21(1a) and Lemma 5.7.5, there exist $b_1, b_2, \beta > 0$ and $d_1, d_2, \delta > 0$ such that for all $\lambda \in [0,1]$, $z \in R + i\mathbb{R}$ and $u \in \mathbb{R}_-$

$$\mathrm{Re}\left(\kappa^\lambda(z)\right) \leq d_1 - d_2 |\mathrm{Im}(z)|^\delta \quad \text{and} \quad \mathrm{E}\left(e^{uV_T}\right) \leq e^{b_1 - b_2|u|^\beta}.$$

Hence, for all $z \in R + i\mathbb{R}$ such that $d_1 - d_2 |\mathrm{Im}(z)|^\delta \leq 0$ and all $\lambda \in [0,1]$

$$\mathrm{E}\left(e^{4\mathrm{Re}\left(\psi^1(\lambda,z)\right)} \right) \leq \exp\left(b_1 - (4\lambda)^\beta \lambda b_2 \left| d_1 - d_2 |\mathrm{Im}(z)|^\delta \right|^\beta \right).$$

Intuitively speaking, we hence obtain that $e^{\mathrm{Re}\left(\psi^0(\lambda,z)\right)}$ decays fast if $\lambda \approx 0$, and $\mathrm{E}\left(e^{4\mathrm{Re}\left(\psi^1(\lambda,z)\right)} \right)$ decays fast if $\lambda \approx 1$. Combining these observations, we obtain for all $\lambda \in [0,1]$ and all $z \in R + i\mathbb{R}$ such that $b_1 - 2^{2-\beta} b_2 \left| d_1 - d_2 |\mathrm{Im}(z)|^\delta \right|^\beta \leq 0$ and such that $d_1 - d_2 |\mathrm{Im}(z)|^\delta \leq 0$

$$\begin{aligned} e^{\mathrm{Re}\left(\psi^0(\lambda,z)\right)} \mathrm{E}\left(e^{4\mathrm{Re}\left(\psi^1(\lambda,z)\right)} \right)^{\frac{1}{4}} \leq& \exp\left(\left(d_1 - d_2 \left| \mathrm{Im}(z)^\delta \right| \right) \left(\frac{1}{2} \mathrm{E}(V_T) + \frac{3}{4} \frac{\mathrm{Var}(M_T)}{\mathrm{Var}(L_1)} \right) \right) \\ &+ \exp\left(\frac{1}{4} \left(b_1 - 2^{2-\beta} b_2 \left| d_1 - d_2 |\mathrm{Im}(z)|^\delta \right|^\beta \right) \right). \end{aligned}$$

Hence, the right-hand side of (5.88) decays exponentially in $|\mathrm{Im}(z)|$ if $|\mathrm{Im}(z)|$ is large enough. By this fact and the estimates obtained in (5.80) and (5.81) to handle the case if $|\mathrm{Im}(z)|$ is small, we see that $(\lambda, z) \mapsto \left|\frac{\partial^n}{\partial \lambda^n}\varphi_{X_T^\lambda}(z)\right|$ is a bounded function on $[0,1]\times(R+i\mathbb{R})$. Since $x \mapsto |p(R+ix)|$ is integrable, setting $M_n(z) = \sup_{\lambda\in[0,1], z\in R+i\mathbb{R}} \left|\frac{\partial^n}{\partial \lambda^n}\varphi_{X_T^\lambda}(z)\right|$ in this case completes the proof. □

5.7.3.7. Integral representation of cash greeks in the Black-Scholes model S^0

Lemma 5.7.14. *$D_n(s)$ in Definition 5.4.2 can be written as*

$$D_n(s) = \int_{R-i\infty}^{R+i\infty} \left(\prod_{i=0}^{n-1}(z-i)\right) s^z e^{\frac{1}{2}\overline{\sigma}^2 z(z-1)T}\, p(z)\,\mathrm{d}z$$

for any $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$, with $p : (R+i\mathbb{R}) \to \mathbb{C}$ from Proposition 5.7.10.

PROOF. Cf. Lemma 3.4.3. □

5.7.3.8. Main theorem

The following proposition rephrases our main Theorem 5.4.3.

Proposition 5.7.15. *For the option price c^λ relative to the price process S^λ, the mapping $\lambda \mapsto c^\lambda$ is in $C^\infty([0,1],\mathbb{R}_+)$. In particular, we have for $\lambda \mapsto c^\lambda$ and its first two derivatives in zero*

$$\begin{aligned} c^0 &= \mathfrak{A}_0(c)\,, \\ \frac{\partial}{\partial\lambda}c^\lambda\bigg|_{\lambda=0} &= \mathfrak{A}_1(c)\,, \\ \frac{\partial^2}{\partial\lambda^2}c^\lambda\bigg|_{\lambda=0} &= \mathfrak{A}_2(c) \end{aligned}$$

with $\mathfrak{A}_0(c)$, $\mathfrak{A}_1(c)$, and $\mathfrak{A}_2(c)$ as in Theorem 5.4.3.

PROOF. The proof is a combination of the statements shown above: by Proposition 5.7.11, for all $\lambda \in [0,1]$ the option price c^λ from (5.22) admits the representation

$$c^\lambda = \int_{R-i\infty}^{R+i\infty} \varphi_{X_T^\lambda}(z)\,p(z)\,\mathrm{d}z \tag{5.89}$$

with $p : (R+i\mathbb{R}) \to \mathbb{C}$ from Proposition 5.7.10 and the extended characteristic function $\varphi_{X_T^\lambda}$ of X_T^λ. By Lemma 5.7.13(i), for all $z \in R+i\mathbb{R}$ we have that $\varphi_{X_T^\lambda}(z)$ is infinitely often differentiable with respect to λ, and (iii) of the same lemma provides the existence of majorants for all derivatives. Corollary C.0.3 then implies that integration with respect to z and differentiation with respect to λ

of arbitrary order can be interchanged, which yields that $\lambda \mapsto c^\lambda$ is in $C^\infty([0,1],\mathbb{R}_+)$. In particular, we obtain

$$c^0 = \int_{R-i\infty}^{R+i\infty} \varphi_{X_T^0}(z)\, p(z)\,\mathrm{d}z,$$

$$\frac{\partial}{\partial\lambda} c^\lambda \bigg|_{\lambda=0} = \int_{R-i\infty}^{R+i\infty} \frac{\partial}{\partial\lambda} \varphi_{X_T^\lambda}(z)\bigg|_{\lambda=0} p(z)\,\mathrm{d}z, \tag{5.90}$$

$$\frac{\partial^2}{\partial\lambda^2} c^\lambda \bigg|_{\lambda=0} = \int_{R-i\infty}^{R+i\infty} \frac{\partial^2}{\partial\lambda^2} \varphi_{X_T^\lambda}(z)\bigg|_{\lambda=0} p(z)\,\mathrm{d}z, \tag{5.91}$$

and we dispose of explicit representations for $\varphi_{X_T^0}$ and the first two derivatives by Lemma 5.7.13(ii). Note that

$$\varphi_{X_T^0}(z) = S_0^z \exp\left(\frac{1}{2}\left(\mathrm{Var}\left(L_1\right)\mathrm{E}\left(V_T\right) + \mathrm{Var}\left(M_T\right)\right) z(z-1)\right) = S_0^z \exp\left(\frac{1}{2}\overline{\sigma} T z(z-1)\right)$$

is the extended characteristic function of $\log(S_T^0)$ for the limiting Black-Scholes process S^0, cf. Proposition 5.2.20. Lemma 5.7.14 implies that c^0 is indeed the Black-Scholes price $C(S_0)$ with function C from (5.29). It remains to treat the integrals (5.90), (5.91) over the first two derivatives of $\varphi_{X_T^\lambda}$. The moments of L_1, V_T and M_T appearing in $\mathfrak{A}_1(c)$, $\mathfrak{A}_2(c)$ from Theorem 5.4.3 come into play by the representations of $\frac{\partial}{\partial\lambda}\varphi_{X_T^\lambda}(z)\Big|_{\lambda=0}$, $\frac{\partial^2}{\partial\lambda^2}\varphi_{X_T^\lambda}(z)\Big|_{\lambda=0}$ from Lemma 5.7.13(ii). By Lemma 5.7.14, the integrals of the form

$$\int_{R-i\infty}^{R+i\infty} \varphi_{X_T^0}(z) r(z)\, p(z)\,\mathrm{d}z,$$

where $r(z)$ is a polynomial in z, are indeed linear combinations of cash greeks as in Definition 5.4.2. To this end, note that for all $n \in \mathbb{N}_{\geq 1}$ there exist $a_1, \dots, a_n \in \mathbb{N}$ such that

$$z^n = \sum_{k=1}^{n} a_k q_k(z),$$

where

$$q_k(z) := \prod_{i=0}^{k-1} (z-i).$$

I.e., every monomial z can be rewritten as a linear combination of polynomials that appear in the integral representation of Black-Scholes cash greeks from Lemma 5.7.14. Rearranging the resulting expressions finally yields the assertion. □

5.7.3.9. Approximation to implied volatility

PROOF OF THEOREM 5.4.8. Since $\widehat{C}'(\sigma) \neq 0$ for all $\sigma > 0$, $\widehat{C}$ is invertible on its image, and the implicit function theorem (cf., e.g., [Rud64, Theorem 9.28]) implies that for its inverse $\widehat{C}^{-1}$: $\widehat{C}((0,\infty)) \to (0,\infty)$ the first derivative is given by

$$\left(\widehat{C}^{-1}\right)'(y) = \frac{1}{\widehat{C}'\left(\widehat{C}^{-1}(y)\right)}, \quad y \in \widehat{C}((0,\infty)). \tag{5.92}$$

Since $\widehat{C}$ is infinitely often differentiable by Lemma 5.4.7, Representation (5.92) yields that $\widehat{C}^{-1}$: $\widehat{C}((0,\infty)) \to (0,\infty)$ is infinitely often differentiable as well. In particular,

$$\left(\widehat{C}^{-1}\right)''(y) = -\frac{\widehat{C}''\left(\widehat{C}^{-1}(y)\right)}{\left(\widehat{C}'\left(\widehat{C}^{-1}(y)\right)\right)^3}, \quad y \in \widehat{C}((0,\infty)). \tag{5.93}$$

Note that $\widehat{C}^{-1}(c^0) = \overline{\sigma}$ since c^0 is the discounted Black-Scholes price of the option with payoff function f and maturity T relative to the volatility parameter $\overline{\sigma}$, cf. Theorem 5.4.3. Recalling that $\mathfrak{A}_1(c) = \frac{\partial}{\partial\lambda}c^\lambda\Big|_{\lambda=0}$ and $\mathfrak{A}_2(c) = \frac{\partial^2}{\partial\lambda^2}c^\lambda\Big|_{\lambda=0}$ by Definition 5.2.28, we see that (5.92), (5.93) and the chain rule yield the assertions for $\mathfrak{A}_1(\sigma_{\text{impl}}) = \frac{\partial}{\partial\lambda}\widehat{C}^{-1}(c^\lambda)\Big|_{\lambda=0}$ and $\mathfrak{A}_2(\sigma_{\text{impl}}) = \frac{\partial^2}{\partial\lambda^2}\widehat{C}^{-1}(c^\lambda)\Big|_{\lambda=0}$ from (5.32). By the integral representation of f from Proposition 5.7.10 and Theorem 3.1.1, we have

$$\widehat{C}(\sigma) = \int_{R-i\infty}^{R+i\infty} S_0^z e^{\frac{1}{2}\sigma^2 z(z-1)T}\, p(z)\,\mathrm{d}z, \quad \sigma > 0.$$

By the same arguments as in the proof of Lemmas 3.4.1 and 3.4.3, we may interchange differentiation with respect to σ and integration with respect to z. Hence,

$$\begin{aligned}
\widehat{C}'(\sigma) &= \int_{R-i\infty}^{R+i\infty} \sigma T z(z-1) S_0^z e^{\frac{1}{2}\sigma^2 z(z-1)T}\, p(z)\,\mathrm{d}z,\\
\widehat{C}''(\sigma) &= \int_{R-i\infty}^{R+i\infty} \left(Tz(z-1) + \sigma^2 T^2 z^2 (z-1)^2\right) S_0^z e^{\frac{1}{2}\sigma^2 z(z-1)T}\, p(z)\,\mathrm{d}z
\end{aligned}$$

for $\sigma > 0$. The integral representation of cash greeks in the Black-Scholes model S^0 from Lemma 5.7.14 then yields the relations from (5.33), which completes the proof. □

5.7.4. Regularity conditions in the Heston model

PROOF OF LEMMA 5.5.1. For all $t \in \mathbb{R}_+$, denote $V_t := \int_0^t y_s\,\mathrm{d}s$. Since $V_t \geq 0$ (cf. also the proof of Lemma 5.5.3), we obtain by dominated convergence

$$\mathrm{E}\left(\lim_{t\to\infty} e^{-V_t}\right) = \lim_{t\to\infty} \mathrm{E}\left(e^{-V_t}\right).$$

Proposition 5.7.16 below provides an explicit representation of $\mathrm{E}\left(e^{-V_t}\right)$. For all $t \in \mathbb{R}_+$, we immediately see that for the functions B_t, γ, g_t defined there, we have

$$\begin{aligned}
B_t(1) &\leq 0,\\
\gamma(1) &> \kappa,\\
g_t(1) &> \frac{\kappa}{\gamma(1)} \exp\left(\frac{\gamma(1)t}{2}\right).
\end{aligned}$$

Thus, we have for the function A_t defined as well in Proposition 5.7.16

$$\begin{aligned} A_t(1) &= \exp\left(\frac{\kappa^2\eta t}{\theta^2} - \frac{2\kappa\eta}{\theta^2}\log\left(g_t(1)\right)\right) \\ &\leq \exp\left(\frac{\kappa^2\eta t}{\theta^2} - \frac{2\kappa\eta}{\theta^2}\left(\log\left(\frac{\kappa}{\gamma(1)}\right) + \frac{\gamma(1)t}{2}\right)\right) \\ &= \exp\left(-\frac{2\kappa\eta}{\theta^2}\left(\log\left(\frac{\kappa}{\gamma(1)}\right) + \frac{(\gamma(1)-\kappa)t}{2}\right)\right). \end{aligned}$$

Hence, we may conclude

$$\lim_{t\to\infty} \mathrm{E}\left(e^{-V_t}\right) = \lim_{t\to\infty}\left(A_t(1)e^{B_t(1)y_0}\right) \leq \lim_{t\to\infty} A_t(1) = 0,$$

recalling that $\kappa, \eta, \theta > 0$. This implies $\lim_{t\to\infty} e^{-V_t} = 0$ almost surely, which yields the assertion. □

PROOF OF LEMMA 5.5.3. It is well-known that for all $t \in \mathbb{R}_+$, the random variable y_t is non-negative and follows a non-central chi squared distribution (cf., e.g., [BM06, Section 3.2.3]), which is a continuous distribution. Hence, for all $t \in \mathbb{R}_+$, we have $1_{\{0\}}(y_t) = 0$ almost surely, and consequently by Fubini's Theorem,

$$\mathrm{E}\left(\int_0^\infty 1_{\{0\}}(y_t)\,\mathrm{d}t\right) = \int_0^\infty \mathrm{E}\left(1_{\{0\}}(y_t)\right)\,\mathrm{d}t = 0.$$

Hence, almost surely, the mapping $s \mapsto y_s$ is non-negative and vanishes only on a Lebesgue null set. This implies that $t \mapsto \int_0^t y_s\,\mathrm{d}s$ is almost surely strictly increasing. □

PROOF OF LEMMA 5.5.5. First, note that in the case of the Heston model, $\exp(U^\lambda) = \mathscr{E}(\lambda M)$ with M from (5.36) for all $\lambda \in [0,1]$. One directly verifies that for all $\lambda \in [0,1]$, the process $(y, \lambda M)$ with y from (5.36) is a semimartingale with affine differential characteristics relative to admissible Lévy-Khintchine triplets

$$\begin{aligned} &\left(\begin{pmatrix}\kappa\eta \\ 0\end{pmatrix}, 0, 0\right), \\ &\left(\begin{pmatrix}-\kappa \\ 0\end{pmatrix}, \begin{pmatrix}\theta^2 & \lambda\theta\rho \\ \lambda\theta\rho & \lambda^2\rho^2\end{pmatrix}, 0\right), \\ &(0,0,0) \end{aligned}$$

in the sense of [Kal06, Definition 3.1/Theorem 3.2]. For all $\lambda \in [0,1]$, the process λM is a local martingale, and hence also $\mathscr{E}(\lambda M)$ is a local martingale since M is continuous. [KMK10, Corollary 3.2] directly implies that $\mathscr{E}(\lambda M)$ is even a martingale, which completes the proof. □

Proposition 5.7.16 (Laplace transform of integrated square root process)**.** *For all $t \in \mathbb{R}_+$, the Laplace transform of $V_t = \int_0^t y_s\,ds$ from* (5.36) *is given by*

$$\mathrm{E}\left(e^{-uV_t}\right) = A_t(u)e^{B_t(u)y_0}, \quad u \in \mathbb{R}_+,$$

for functions $A_t, B_t : \mathbb{R}_+ \to \mathbb{R}$ with

$$
\begin{aligned}
A_t(u) &= \frac{\exp\left(\frac{\kappa^2 \eta t}{\theta^2}\right)}{\exp\left(\frac{2\kappa\eta}{\theta^2}\log(g_t(u))\right)},\\
B_t(u) &= \frac{-2u}{\kappa + \gamma(u)\coth\left(\frac{\gamma(u)t}{2}\right)},\\
\gamma(u) &= \sqrt{\kappa^2 + 2\theta^2 u},\\
g_t(u) &= \cosh\left(\frac{\gamma(u)t}{2}\right) + \frac{\kappa}{\gamma(u)}\sinh\left(\frac{\gamma(u)t}{2}\right).
\end{aligned}
$$

PROOF OF PROPOSITION 5.7.16. Cf. [EK05, Theorem 9.6.4], and represent the expressions there in terms of hyperbolic trigonometric functions. □

PROOF OF LEMMA 5.5.6. The assertion is implied directly by Proposition 5.7.16, using that for the functions A_t, B_t, γ, g_t defined there and $u \in \mathbb{R}_+$, we have

$$
\begin{aligned}
\gamma(u) &> \theta\sqrt{2}\sqrt{u},\\
g_t(u) &> \frac{1}{2}\exp\left(\frac{\gamma(u)t}{2}\right),\\
B_t(u) &\leq 0.
\end{aligned}
$$
□

PROOF OF LEMMA 5.5.8. We know from the proof of Lemma 5.5.5 that (y, M) with y and M from (5.36) is a semimartingale with affine differential characteristics relative to admissible Lévy-Khintchine triplets

$$
\begin{aligned}
&\left(\begin{pmatrix}\kappa\eta\\ 0\end{pmatrix}, 0, 0\right),\\
&\left(\begin{pmatrix}-\kappa\\ 0\end{pmatrix}, \begin{pmatrix}\theta^2 & \theta\rho\\ \theta\rho & \rho^2\end{pmatrix}, 0\right),\\
&(0,0,0).
\end{aligned}
$$

For fixed $t > 0$ and $p \in \mathbb{R}$, let us consider the related system of Riccati equations

$$
\begin{aligned}
\frac{\partial}{\partial s}\psi^0(s) &= \kappa\eta\,\psi^1(s),\\
\psi^0(0) &= 0,\\
\frac{\partial}{\partial s}\psi^1(s) &= \frac{1}{2}\begin{pmatrix}\psi^1(s)\\ \psi^2(s)\end{pmatrix}^\top \begin{pmatrix}\theta^2 & \theta\rho\\ \theta\rho & \rho^2\end{pmatrix}\begin{pmatrix}\psi^1(s)\\ \psi^2(s)\end{pmatrix} + \begin{pmatrix}-\kappa\\ 0\end{pmatrix}^\top \begin{pmatrix}\psi^1(s)\\ \psi^2(s)\end{pmatrix},\\
\psi^1(0) &= 0,\\
\frac{\partial}{\partial s}\psi^2(s) &= 0,\\
\psi^2(s) &= p,
\end{aligned}
$$

where we look for solutions $\psi^0, \psi^1, \psi^2 \in C^1([0,t])$. We directly see that $\psi^2(s) = p$ for all $s \in [0,t]$, and $\psi^0(s) = \kappa\eta \int_0^s \psi^1(u)\,\mathrm{d}u$ once we have found a solution for ψ^1. Using $\psi^2 \equiv p$, the condition on ψ^1 reduces to

$$\begin{aligned}\frac{\partial}{\partial s}\psi^1(s) &= \frac{1}{2}\theta^2\psi^1(s)^2 + (\theta\rho p - \kappa)\psi^1(s) + \frac{1}{2}\rho^2 p^2,\\ \psi^1(0) &= 0.\end{aligned}$$

This ODE allows for a solution on $[0,t]$ if $\rho p < \frac{\kappa}{2\theta}$, as one can conclude, e.g., from the proof of [FM09, Lemma 5.2]. In this situation, [KMK10, Theorem 5.1] yields

$$\mathrm{E}\left(e^{pM_t}\right) = e^{\psi^0(t)} < \infty,$$

which completes the proof of the first assertion.

For the second part of the assertion, we could employ similar techniques since (y, M, U) is an affine process as well. However, we waive with a more restrictive condition that we obtain from a simple Hölder-type argument. For all $t \in \mathbb{R}_+$ and all $b \in \mathbb{R}$, we have by Hölder's inequality

$$\mathrm{E}\left(e^{bU_t}\right)^2 = \mathrm{E}\left(e^{bM_t - b\frac{1}{2}\rho^2 V_t}\right)^2 \leq \mathrm{E}\left(e^{2bM_t}\right)\mathrm{E}\left(e^{-b\rho^2 V_t}\right).$$

By the first part of the assertion and by Lemma 5.5.7, both terms on the right-hand side are finite if $2b\rho < \frac{\kappa}{2\theta}$ and $-b\rho^2 \leq \frac{\kappa^2}{2\theta^2}$, which completes the proof. □

5.7.5. Regularity conditions in the extended Stein & Stein model

Proposition 5.7.17 (Characteristic function of integrated squared Gaussian OU process)**.** *For all $t \in \mathbb{R}_+$, the characteristic function of the integrated squared Gaussian OU process $V_t = \int_0^t y_s^2\,\mathrm{d}s$ with the OU process y from* (5.38) *is given by*

$$\varphi_{V_t}(u) = \mathrm{E}\left(e^{iuV_t}\right) = e^{\frac{1}{2}\left(D_t(u)y_0^2 + B_t(u)y_0 + C_t(u)\right)}, \quad u \in \mathbb{R},$$

for functions D_t, B_t, C_t, g_t, h_t, $\gamma : \mathbb{R} \to \mathbb{C}$ given by

$$D_t(u) = \frac{1}{\theta^2}\left(\kappa - \gamma(u)\frac{g_t(u)}{h_t(u)}\right), \tag{5.94}$$

$$B_t(u) = \frac{1}{\theta^2\gamma(u)}\left(\frac{\kappa\eta\gamma(u) - \frac{\kappa^3\eta}{\gamma(u)} + \kappa^2\eta g_t(u)}{h_t(u)} - \kappa\eta\gamma(u)\right), \tag{5.95}$$

$$\begin{aligned} C_t(u) = &-\frac{1}{2}\mathrm{Log}\frac{h_t}{\exp(\kappa t)}(u) + \frac{\kappa^2\eta^2\gamma(u)^2 - \kappa^4\eta^2}{2\theta^2\gamma(u)^3}\left(\frac{\sinh(\gamma(u)t)}{h_t(u)} - \gamma(u)t\right) \\ &+ \frac{\kappa^3\eta^2\gamma(u) - \frac{\kappa^5\eta^2}{\gamma(u)}}{\theta^2\gamma(u)^3}\left(\frac{\cosh(\gamma(u)t) - 1}{h_t(u)}\right), \end{aligned} \tag{5.96}$$

$$g_t(u) = \sinh(\gamma(u)t) + \frac{\kappa}{\gamma(u)}\cosh(\gamma(u)t), \tag{5.97}$$

$$h_t(u) = \cosh(\gamma(u)t) + \frac{\kappa}{\gamma(u)}\sinh(\gamma(u)t), \tag{5.98}$$

$$\gamma(u) = \sqrt{\kappa^2 - 2\theta^2 iu}, \tag{5.99}$$

where Log *denotes the distinguished logarithm in the sense of [Sat99, Lemma 7.6], and $\sqrt{\cdot}$ is the analytic extension of the real square root to $\mathbb{C}\backslash\mathbb{R}_-$.*

PROOF. Cf. [SZ99, Appendix A]. □

PROOF OF LEMMA 5.5.10. By the same reasoning as in the proof of Lemma 5.5.1, is is sufficient to show that $\lim_{t\to\infty} \mathrm{E}\left(e^{-V_t}\right) = 0$. To this end, we use the representation of $\mathrm{E}\left(e^{-V_t}\right)$ from Lemma 5.7.18 below. For the functions from (5.100)–(5.104), we see that $\overline{\gamma}(-1) > \kappa$, $\overline{h}_t(-1) \geq 1$, and $\overline{g}_t(-1) \geq 0$ for all $t \in \mathbb{R}_+$. Moreover, one easily verifies

$$\frac{\overline{g}_t(-1)}{\overline{h}_t(-1)} \leq \frac{\overline{\gamma}(-1)}{\kappa}, \quad t \in \mathbb{R}_+,$$

and

$$\lim_{t\to\infty} \overline{h}_t(-1) = \infty.$$

These facts and the continuity of $t \mapsto \overline{g}_t(-1)$ on $\mathbb{R}_+$ imply that $t \mapsto \overline{D}_t(-1)$, $t \mapsto \overline{B}_t(-1)$ are bounded from above on $\mathbb{R}_+$. In order to treat function $\overline{C}_t(-1)$, observe that

$$-\frac{1}{2}\log\left(\frac{\overline{h}_t(-1)}{\exp(\kappa t)}\right) \leq -\frac{1}{2}\log\left(\frac{\exp(\overline{\gamma}(-1)t)}{2\exp(\kappa t)}\right) \leq -\frac{1}{2}\log\left(\frac{1}{2}\right)$$

for all $t \in \mathbb{R}_+$. Moreover, one easily sees that

$$0 \leq \frac{\sinh(\overline{\gamma}(-1)t)}{\overline{h}_t(-1)} \leq \frac{\overline{\gamma}(-1)}{\kappa},$$

and

$$0 \leq \frac{\cosh(\overline{\gamma}(-1)t) - 1}{\overline{h}_t(-1)} \leq \frac{\cosh(\overline{\gamma}(-1)t) - 1}{\cosh(\overline{\gamma}(-1)t)} \leq 1 - 2\exp(-\overline{\gamma}(-1)t) \leq 2$$

for all $t \in \mathbb{R}_+$. This implies that $t \mapsto \overline{C}_t(-1) + \frac{\kappa^2\eta^2\overline{\gamma}(-1)^2-\kappa^4\eta^2}{2\theta^2\overline{\gamma}(-1)^2}t$ is bounded from above on $\mathbb{R}_+$. However,

$$\lim_{t\to\infty} -\frac{\kappa^2\eta^2\overline{\gamma}(-1)^2-\kappa^4\eta^2}{2\theta^2\overline{\gamma}(-1)^2}t = -\infty$$

since $\overline{\gamma}(-1) > \kappa^2$. Inserting these estimates in the representation of $\mathrm{E}\left(e^{-V_t}\right)$ from Lemma 5.7.18 yields $\lim_{t\to\infty}\mathrm{E}\left(e^{-V_t}\right) = 0$, which completes the proof. □

PROOF OF LEMMA 5.5.11. The assertion follows along the same lines as the proof of Lemma 5.5.3. However, one uses that for all $t \in \mathbb{R}_+$, the random variable y_t follows a normal distribution, and hence y_t^2 is non-negative and continuously distributed. □

Lemma 5.7.18 (Moment generating function of integrated squared Gaussian OU process)**.** *For all $t \in \mathbb{R}_+$, the moment generating function of the integrated squared Gaussian OU process $V_t = \int_0^t y_s^2\,\mathrm{d}s$ with the OU process y from* (5.38) *is finite on $(-\infty, \frac{\kappa^2}{2\theta^2})$ and given by*

$$m_{V_t}(u) = \mathrm{E}\left(e^{uV_t}\right) = e^{\frac{1}{2}\left(\overline{D}_t(u)y_0^2+\overline{B}_t(u)y_0+\overline{C}_t(u)\right)}, \quad u < \frac{\kappa^2}{2\theta^2},$$

for functions $\overline{D}_t$, $\overline{B}_t$, $\overline{C}_t$, $\overline{g}_t$, $\overline{h}_t$, $\overline{\gamma}\colon (-\infty, \frac{\kappa^2}{2\theta^2}) \to \mathbb{R}$ given by

$$\overline{D}_t(u) = \frac{1}{\theta^2}\left(\kappa - \overline{\gamma}(u)\frac{\overline{g}_t(u)}{\overline{h}_t(u)}\right), \tag{5.100}$$

$$\overline{B}_t(u) = \frac{1}{\theta^2\overline{\gamma}(u)}\left(\frac{\kappa\eta\overline{\gamma}(u) - \frac{\kappa^3\eta}{\overline{\gamma}(u)} + \kappa^2\eta\overline{g}_t(u)}{\overline{h}_t(u)} - \kappa\eta\overline{\gamma}(u)\right), \tag{5.101}$$

$$\begin{aligned}\overline{C}_t(u) = &-\frac{1}{2}\log\left(\frac{\overline{h}_t(u)}{\exp(\kappa t)}\right) + \frac{\kappa^2\eta^2\overline{\gamma}(u)^2-\kappa^4\eta^2}{2\theta^2\overline{\gamma}(u)^3}\left(\frac{\sinh(\overline{\gamma}(u)t)}{\overline{h}_t(u)} - \overline{\gamma}(u)t\right)\\ &+\frac{\kappa^3\eta^2\overline{\gamma}(u) - \frac{\kappa^5\eta^2}{\overline{\gamma}(u)}}{\theta^2\overline{\gamma}(u)^3}\left(\frac{\cosh(\overline{\gamma}(u)t)-1}{\overline{h}_t(u)}\right),\end{aligned} \tag{5.102}$$

$$\overline{g}_t(u) = \sinh(\overline{\gamma}(u)t) + \frac{\kappa}{\overline{\gamma}(u)}\cosh(\overline{\gamma}(u)t),$$

$$\overline{h}_t(u) = \cosh(\overline{\gamma}(u)t) + \frac{\kappa}{\overline{\gamma}(u)}\sinh(\overline{\gamma}(u)t), \tag{5.103}$$

$$\overline{\gamma}(u) = \sqrt{\kappa^2 - 2\theta^2 u}. \tag{5.104}$$

PROOF OF LEMMA 5.7.18. We employ an analytic extension argument to the characteristic function of V_t. Set $\Lambda := \left\{z \in \mathbb{C} : -\mathrm{Im}(z) < \frac{\kappa^2}{2\theta^2}\right\}$. Then, for all $t \in \mathbb{R}_+$, the characteristic function φ_{V_t} of V_t given by Proposition 5.7.17 allows for an analytic extension to Λ: by the choice of $\sqrt{\cdot}$, function γ from (5.99) can be extended analytically to $\widetilde{\gamma}$ on Λ since $\mathrm{Re}\left(\kappa^2 - 2\theta^2 iz\right) > 0$ for all $z \in \Lambda$. Additionally, $\mathrm{Re}(\widetilde{\gamma}(z)) > 0$ for all $z \in \Lambda$, in particular $\widetilde{\gamma} \neq 0$ on Λ. Moreover, h_t and g_t from (5.98) and (5.97) allow for extensions $\widetilde{h}_t$, $\widetilde{g}_t$ to Λ, and $\widetilde{h}_t \neq 0$ on Λ. In order to see this, note that $\widetilde{h}_t(z) = 0$ is equivalent to $-\frac{\widetilde{\gamma}(z)}{\kappa} = \tanh(\widetilde{\gamma}(z)t)$. Taking the real part of both sides yields

$$-\frac{\mathrm{Re}(\widetilde{\gamma}(z))}{\kappa} = \frac{\cosh(2\mathrm{Re}(\widetilde{\gamma}(z))t)}{\cosh(2\mathrm{Re}(\widetilde{\gamma}(z))t) + \cos(2\mathrm{Im}(\widetilde{\gamma}(z))t)}.$$

For $z \in \Lambda$, we have $\operatorname{Re}(\widetilde{\gamma}(z)) > 0$, and hence the right-hand side is positive, while the left-hand side is negative. This contradiction shows that $\widetilde{h}_t \neq 0$ on Λ. Finally, let us check that also $\operatorname{Log}\frac{h_t}{\exp(\kappa t)}$ allows for an analytic extension to Λ. Since $\widetilde{h}_t \neq 0$ on Λ, which is an open and convex set, by [FL94, Satz V.1.4], there exists an analytic function $l : \Lambda \to \mathbb{C}$ such that $e^{l(z)} = \frac{\widetilde{h}_t(z)}{\exp(\kappa t)}$ for all $z \in \Lambda$. In particular,

$$e^{l(u)} = \frac{\widetilde{h}_t(u)}{\exp(\kappa t)} = \frac{h_t(u)}{\exp(\kappa t)} = e^{\operatorname{Log}\frac{h_t}{\exp(\kappa t)}(u)}$$

for all $u \in \mathbb{R}$. Hence, $l(u) - \operatorname{Log}\frac{h_t}{\exp(\kappa t)}(u) = 2\pi i k(u)$, $u \in \mathbb{R}$, for a function $k : \mathbb{R} \to \mathbb{Z}$. However, by the continuity of l and $\operatorname{Log}\frac{h_t}{\exp(\kappa t)}$, function k must be constant. Hence, $j := l - 2\pi i k(0)$ is the desired extension of $\operatorname{Log}\frac{h_t}{\exp(\kappa t)}$. The above considerations are the essential points to see that functions D_t, B_t, C_t from (5.94), (5.95), (5.96) allow for analytic extensions $\widetilde{D}_t$, $\widetilde{B}_t$, $\widetilde{C}_t$ to Λ, and hence so does φ_{V_t}, where its extension $\widetilde{\varphi}_{V_t}$ is given by

$$\widetilde{\varphi}_{V_t}(z) = e^{\frac{1}{2}\left(\widetilde{D}_t(z)y_0^2 + \widetilde{B}_t(z)y_0 + \widetilde{C}_t(z)\right)}, \quad z \in \Lambda.$$

By [DFS03, Lemmas A.2 and A.4], we can draw the essential conclusion that

$$\mathrm{E}\left(e^{aV_t}\right) = \widetilde{\varphi}_{V_t}(-ia), \quad a < \frac{\kappa^2}{2\theta^2}.$$

Note that for the functions defined in (5.100)–(5.104), we have $\overline{D}_t(a) = \widetilde{D}_t(-ia)$, $\overline{B}_t(a) = \widetilde{B}_t(-ia)$, etc. for $a < \frac{\kappa^2}{2\theta^2}$. Here, we have to take care that we can really use the real-valued logarithm in function $\overline{C}_t$. By definition, $e^{j(-ia)} = \frac{\widetilde{h}_t(-ia)}{\exp(\kappa t)} = \frac{\overline{h}_t(a)}{\exp(\kappa t)} \in \mathbb{R}$, $a < \frac{\kappa^2}{2\theta^2}$, and hence $j(-ia) = \log\left(\frac{\overline{h}_t(a)}{\exp(\kappa t)}\right) + 2\pi i\overline{k}(a)$ for some function $\overline{k} : \left(-\infty, \frac{\kappa^2}{2\theta^2}\right) \to \mathbb{Z}$. By the continuity of j and $\log\left(\frac{\overline{h}_t}{\exp(\kappa t)}\right)$, $\overline{k}$ must be constant. Moreover,

$$2\pi i\overline{k}(0) = j(0) - \log\left(\frac{\overline{h}_t(0)}{\exp(\kappa t)}\right) = j(0) = \operatorname{Log}\frac{h_t}{\exp(\kappa t)}(0) = 0,$$

where the last equality holds by construction of the distinguished logarithm. Hence, $\overline{k} \equiv 0$, i.e., $j(-ia) = \log\left(\frac{\overline{h}_t(a)}{\exp(\kappa t)}\right)$ for all $a < \frac{\kappa^2}{2\theta^2}$, which completes the proof. □

Proof of Lemma 5.5.13. Note that in the present situation, $e^{U^\lambda} = \mathscr{E}(\lambda M)$ for the local martingale M from (5.39). It holds $\langle \lambda M, \lambda M \rangle_t = \lambda^2\rho^2 \int_0^t y_s^2 \,\mathrm{d}s = \lambda^2\rho^2 V_t$ for all $t \in \mathbb{R}_+$. Hence, for all $\lambda \in [0,1]$ and all $t \in \mathbb{R}_+$,

$$\mathrm{E}\left(e^{\frac{1}{2}\langle \lambda M, \lambda M \rangle_t}\right) = \mathrm{E}\left(e^{\frac{1}{2}\lambda^2\rho^2 V_t}\right) \leq \mathrm{E}\left(e^{\frac{1}{2}\rho^2 V_t}\right).$$

The right-hand side is finite if $\frac{1}{2}\rho^2 < \frac{\kappa^2}{2\theta^2}$ by Lemma 5.7.18. Hence, if this condition holds, $\mathscr{E}(\lambda M) = e^{U^\lambda}$ is a martingale for all $\lambda \in [0,1]$ by Novikov's criterion (cf. [RY91, Corollary VIII.1.16]), which yields the assertion. □

PROOF OF LEMMA 5.5.14. Fix some $t \in \mathbb{R}_+$. We use the representation of $\mathrm{E}\left(e^{uV_t}\right)$, $u \in \mathbb{R}_-$, from Lemma 5.7.18 to show the assertion. For the functions defined in (5.100)–(5.104), we have $\overline{\gamma}(u) \geq \kappa$, $\overline{g}_t(u) \geq 0$, and $\overline{h}_t(u) \geq 1$ for all $u \in \mathbb{R}_-$, and hence $\overline{D}_t(u) \leq \frac{\kappa}{\theta^2}$ for all $u \in \mathbb{R}_-$. It is easy to see that

$$\lim_{u \to -\infty} \overline{\gamma}(u) = \infty,$$
$$\lim_{u \to -\infty} \overline{h}_t(u) = \infty,$$
$$\lim_{u \to -\infty} \frac{\overline{g}_t(u)}{\overline{h}_t(u)} = 1.$$

This and the continuity of $\overline{\gamma}$, $\overline{g}_t$, $\overline{h}_t$ on $\mathbb{R}_-$ yield that $\overline{B}_t$ is bounded from above on $\mathbb{R}_-$. In order to treat function $\overline{C}_t$, note that

$$\lim_{u \to -\infty} \frac{\sinh(\overline{\gamma}(u)t)}{\overline{h}_t(u)} = 1,$$
$$\lim_{u \to -\infty} \frac{\cosh(\overline{\gamma}(u)t) - 1}{\overline{h}_t(u)} = 1.$$

Then, we see from the continuity of $\overline{\gamma}$, $\overline{h}_t$, cosh, sinh that $\overline{C}_t + \frac{1}{2}\log\left(\frac{\overline{h}_t}{\exp(\kappa t)}\right)$ is bounded from above on $\mathbb{R}_-$. Using that

$$\overline{h}_t(u) \geq \frac{1}{2}\exp(\overline{\gamma}(u)t) \geq \frac{1}{2}\exp(\theta\sqrt{2}t\sqrt{|u|})$$

for all $u \in \mathbb{R}_-$, we obtain

$$-\frac{1}{2}\log\left(\frac{\overline{h}_t(u)}{\exp(\kappa t)}\right) \leq \frac{1}{2}\log(2\exp(\kappa t)) - \frac{1}{2}\theta\sqrt{2}t\sqrt{|u|}$$

for all $u \in \mathbb{R}_-$. Inserting these estimates in the representation of $\mathrm{E}\left(e^{uV_t}\right)$ from Lemma 5.7.18 completes the proof. □

PROOF OF LEMMA 5.5.16. Observe that $\langle M, M \rangle = \rho^2 V$ with M and V from (5.39). For all $t \in \mathbb{R}_+$ and all $a \in \mathbb{R}$, we have by Hölder's inequality

$$\begin{aligned}
\mathrm{E}(\exp(aM_t)) &= \mathrm{E}(\exp(aM_t - \langle aM, aM \rangle_t + \langle aM, aM \rangle_t)) \\
&\leq \mathrm{E}\left(\exp\left(2aM_t - \frac{1}{2}\langle 2aM, 2aM \rangle_t\right)\right)^{\frac{1}{2}} \mathrm{E}(2\langle aM, aM \rangle_t)^{\frac{1}{2}} \\
&= \mathrm{E}(\mathscr{E}(2aM)_t)^{\frac{1}{2}}\, \mathrm{E}\left(\exp(2a^2\rho^2 V_t)\right)^{\frac{1}{2}}.
\end{aligned}$$

The first expectation is finite since $\mathscr{E}(2aM)$ is a positive local martingale and hence a supermartingale. By Lemma 5.7.18, the second expectation is finite if $2a^2\rho^2 < \frac{\kappa^2}{2\theta^2}$, which yields the first assertion. For the second assertion, observe that by Hölder's inequality for all $t \in \mathbb{R}_+$ and $b \in \mathbb{R}$,

$$\mathrm{E}\left(e^{bU_t}\right)^2 = \mathrm{E}\left(e^{bM_t - b\frac{1}{2}\rho^2 V_t}\right)^2 \leq \mathrm{E}\left(e^{2bM_t}\right)\mathrm{E}\left(e^{-b\rho^2 V_t}\right).$$

By the first part of the assertion, the first expectation on the right-hand side is finite if $|\rho b| < \frac{\kappa}{4\theta}$, and by Lemma 5.5.15, the second expectation is finite if $-b\rho^2 < \frac{\kappa^2}{2\theta^2}$, which completes the proof. □

5.7.6. Regularity conditions in the NIG-CIR model

PROOF OF LEMMA 5.5.22. If L is an NIG Lévy process, the corresponding Lévy measure F^L is absolutely continuous with respect to the Lebesgue measure with density

$$\nu_{\mathrm{NIG}}(x) = \frac{\delta\alpha}{\pi} e^{\beta x} \frac{K_1(\alpha|x|)}{|x|}, \quad x \in \mathbb{R}\backslash\{0\},$$

cf. [Sch03, Section 5.3.8]. Here, K_1 denotes the modified Bessel function of the second kind with index 1, cf. [CT03, Appendix A] for more details. By the same reference, we have

$$K_1(|x|) \sim \frac{1}{|x|} \quad \text{as } x \to 0.$$

Hence, for $0 < \varepsilon < 1$, there exists $\delta > 0$ such that for all $0 < r < \delta$, we have

$$\begin{aligned}
\int_{-r}^{r} x^2 F^L(\mathrm{d}x) &= \int_{-r}^{r} x^2 \frac{\delta\alpha}{\pi} e^{\beta x} \frac{K_1(\alpha|x|)}{|x|} \,\mathrm{d}x \\
&\geq \frac{\delta\alpha}{\pi} \int_{-r}^{r} x^2 e^{\beta x} \frac{\frac{1}{\alpha|x|}(1-\varepsilon)}{|x|} \,\mathrm{d}x \\
&= \frac{\delta(1-\varepsilon)}{\pi} \int_{-r}^{r} e^{\beta x} \,\mathrm{d}x \\
&\geq \frac{\delta(1-\varepsilon)}{\pi} e^{-|\beta| r} 2r.
\end{aligned}$$

This implies that for all $0 < \gamma < 1$

$$\lim_{r \to 0} r^{\gamma-2} \int_{-r}^{r} x^2 F^L(\mathrm{d}x) = \infty,$$

hence Condition (5.20) holds. □

5.8. Conclusion

We provide a second-order approximation to the price of a European option in a very general framework for the price process of the underlying, encompassing bivariate stochastic volatility diffusion models, the model class according to Barndorff-Nielsen & Shephard, geometric Lévy, as well as time-changed Lévy models with leverage. Following our general perturbation approach from Chapter 2, the approximation is obtained by considering the complex price process of interest as a perturbed Black-Scholes model: essentially, we connect the Lévy process in the representation of the logarithmic price process to Brownian motion similarly as in Chapter 4 on approximate hedging, and the integrated stochastic volatility process is connected to an appropriate deterministic function. Based on the approximation to the price, we derive a second-order approximation to implied volatility.

Qualitatively, our results show that the deviation of prices in stochastic volatility models with jumps from Black-Scholes prices is essentially determined by the third and fourth moment of the involved Lévy process, the first two moments of integrated stochastic variance, and Black-Scholes sensitivities (cash greeks) of the option. The fine structure of the stock price process is less relevant.

Quantitatively, experiments in four models from the literature for several reasonable parameter choices indicate that – despite the generality of our framework – our approximation leads to very reasonable results, in particular compared to competing, tailor-made approximations.

A. Technical lemmas concerning stochastic calculus

In the following, $\mathbb{D}(\mathbb{R})$ denotes the space of $\mathbb{R}$-valued càdlàg functions on $\mathbb{R}_+$, and $\mathscr{D}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{D}(\mathbb{R})$ relative to the Skorokhod topology, cf. [JS03, Section VI.1].

Lemma A.0.1. *Let M be a positive local martingale defined on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ such that $\mathrm{E}(M_0) = \mathrm{E}(M_t) < \infty$ for all $t \in \mathbb{R}_+$. Then M is a martingale.*

PROOF. By [Jac79, Lemme 5.17], M is a supermartingale, and hence M is adapted and integrable. Fix now $0 \leq s \leq t$, and set $X := M_s - \mathrm{E}(M_t|\mathscr{F}_s)$. Then, $X \geq 0$ almost surely since M is a supermartingale, and $\mathrm{E}(X) = \mathrm{E}(M_s) - \mathrm{E}(M_t) = 0$ by hypothesis. Hence, $X = 0$ almost surely, i.e., $\mathrm{E}(M_t|\mathscr{F}_s) = M_s$ almost surely. □

Lemma A.0.2. *Let M be a martingale on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$, and let $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$ be another filtration such that $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is independent of $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$, i.e., $\mathscr{F}_{\infty-} = \cup_{t\geq 0}\mathscr{F}_t$ is independent of $\mathscr{G}_{\infty-} = \cup_{t\geq 0}\mathscr{G}_t$. Then M is also a martingale with respect to the filtration $(\mathscr{H}_t)_{t\in\mathbb{R}_+}$ given by $\mathscr{H}_t := \sigma(\mathscr{F}_t \cup \mathscr{G}_t)$.*

PROOF. The property that M has càdlàg paths and that $|M_t|$ is integrable for all $t \in \mathbb{R}_+$ does not depend on the filtration. Clearly, M is also adapted to $(\mathscr{H}_t)_{t\in\mathbb{R}_+}$. For $0 \leq s \leq t$, we have that M_t is independent of $\mathscr{G}_s$ since M_t is $\mathscr{F}_t$-measurable, and $\mathscr{F}_t$ is independent of $\mathscr{G}_s$ by assumption. [Bau78, Satz 54.4] then yields

$$\mathrm{E}(M_t|\,\mathscr{H}_s) = \mathrm{E}(M_t|\,\sigma(\mathscr{F}_s \cup \mathscr{G}_s)) = \mathrm{E}(M_t|\,\mathscr{F}_s) = M_s. \qquad \square$$

Lemma A.0.3. *The mapping*

$$g : \mathbb{D}(\mathbb{R}) \times \mathbb{R}_+ \to \mathbb{R}, \quad (\alpha, t) \mapsto \alpha(t),$$

is $\mathscr{D}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}_+) - \mathscr{B}(\mathbb{R})$-measurable.

PROOF. To show that g is measurable, we will show that g is the point-wise limit of the sequence of measurable functions $(g_n)_{n\in\mathbb{N}}$ given by

$$g_n : \mathbb{D}(\mathbb{R}) \times \mathbb{R}_+ \to \mathbb{R}, \quad (\alpha, t) \mapsto n \int_t^{t+\frac{1}{n}} \alpha(s)\mathrm{d}s.$$

For all $\alpha \in \mathbb{D}(\mathbb{R})$ and $t \in \mathbb{R}_+$, we have indeed by the mean value theorem

$$\lim_{n\to\infty} g_n(\alpha, t) = \alpha(t)$$

since α is càdlàg.

Since $\mathbb{D}(\mathbb{R})$ equipped with the topology induced by the Skorokhod metric and $\mathbb{R}_+$ equipped with the topology induced by the standard metric are both Polish spaces, the product space $\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+$ equipped with the corresponding product metric is again a Polish space, cf., e.g., [Bau78, §41]. It is then easy to check that

$$\mathscr{B}(\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+) = \mathscr{D}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}_+).$$

To show that g_n is measurable for all $n \in \mathbb{N}$, it is hence sufficient to show that g_n is continuous with respect the product metric on $\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+$. Hence, let $((\alpha_k, t_k))_{k \in \mathbb{N}}$ be a sequence in $\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+$ convergent to $(\alpha, t) \in \mathbb{D}(\mathbb{R}) \times \mathbb{R}_+$, i.e., $\alpha_k \to \alpha$ in $\mathbb{D}(\mathbb{R})$, and $t_k \to t$ as $k \to \infty$. For all $n, k \in \mathbb{N}$ we have

$$\begin{aligned} |g_n(\alpha_k, t_k) - g_n(\alpha, t)| &\leq n \int \left| 1_{[t_k, t_k + \frac{1}{n}]} - 1_{[t, t + \frac{1}{n}]} \right| |\alpha_k(s)| \, \mathrm{d}s + n \int 1_{[t, t+\frac{1}{n}]} |\alpha_k(s) - \alpha(s)| \, \mathrm{d}s \\ &\leq \sup_{s \leq \tilde{t}} |\alpha_k(s)| \, n \int \left| 1_{[t_k, t_k + \frac{1}{n}]} - 1_{[t, t + \frac{1}{n}]} \right| \mathrm{d}s + n \int 1_{[t, t+\frac{1}{n}]} |\alpha_k(s) - \alpha(s)| \,, \end{aligned} \tag{A.1}$$

where $\tilde{t}$ is chosen such that $\tilde{t} > \max_{n,k \in \mathbb{N}} \left\{ t_k + \frac{1}{n} \vee t + \frac{1}{n} \right\}$ and such that $\tilde{t}$ is not a point of discontinuity of α. Then, [JS03, Proposition VI.2.4] yields

$$\lim_{k \to \infty} \sup_{s \leq \tilde{t}} |\alpha_k(s)| = \sup_{s \leq \tilde{t}} |\alpha(s)| \,.$$

Moreover, for all $n \in \mathbb{N}$,

$$\lim_{k \to \infty} n \int \left| 1_{[t_k, t_k + \frac{1}{n}]} - 1_{[t, t + \frac{1}{n}]} \right| \mathrm{d}s = \lim_{k \to \infty} 2n \, |t_k - t| = 0.$$

Hence, for all $n \in \mathbb{N}$, the first summand in (A.1) goes to 0 as $k \to \infty$. In order to treat the second summand, note that as well by [JS03, Proposition VI.2.4], we find a majorant to apply dominated convergence to the second integral. By the proof of [JS03, Lemma VI.1.44], we have $\lim_{k \to \infty} \alpha_k(s) = \alpha_k(s)$ for all continuity points s of α. However, the set of discontinuity points of α is at most countable, cf. [JS03, VI.1.7]. Hence, for all $n \in \mathbb{N}$,

$$\lim_{k \to \infty} n \int 1_{[t, t+\frac{1}{n}]} |\alpha_k(s) - \alpha(s)| = 0$$

by dominated convergence. Altogether, we obtain

$$\lim_{k \to \infty} |g_n(\alpha_k, t_k) - g_n(\alpha, t)| = 0$$

for all $n \in \mathbb{N}$, which completes the proof. □

Lemma A.0.4. *Let X be an $\mathbb{R}$-valued stochastic process with càdlàg paths on the probability space $(\Omega, \mathscr{F}, P)$, and let V be an $\mathbb{R}_+$-valued random variable such that $X_V \geq 0$ or such that X_V is integrable. Moreover, let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$ such that V is $\mathscr{G}$-measurable and X is independent of $\mathscr{G}$. Then, the mapping*

$$h : \mathbb{R}_+ \to \overline{\mathbb{R}}, \quad t \mapsto \mathrm{E}(X_t) \,,$$

is well defined for P^V-almost all $t \in \mathbb{R}_+$ (i.e., $\mathrm{E}\left(X_t^+\right) < \infty$ or $\mathrm{E}\left(X_t^-\right) < \infty$ for P^V-almost all $t \in \mathbb{R}_+$), and

$$\mathrm{E}\left(X_V \,|\, \mathscr{G}\right) = h(V).$$

PROOF. 1. Let us first make some preliminary considerations. Since X has càdlàg paths, we may consider it as a random variable $X : (\Omega, \mathscr{F}) \to (\mathbb{D}(\mathbb{R}), \mathscr{D}(\mathbb{R}))$. Hence, the $(\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+, \mathscr{D}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}_+))$-valued mapping $\omega \mapsto (X(\omega), V(\omega))^\top$ is as well measurable. Since the function $g : \mathbb{D}(\mathbb{R}) \times \mathbb{R}_+ \to \mathbb{R}, (\alpha, t) \mapsto \alpha(t)$ is measurable by Lemma A.0.3, $X_V = g((X,V))$ is $\mathscr{F} - \mathscr{B}(\mathbb{R})$-measurable as well.

2. Let us know consider the case $X_V \geq 0$. By Fubini's Theorem, the mapping $t \mapsto \mathrm{E}\left(X_t^-\right)$ is $\mathscr{B}(\mathbb{R}_+) - \mathscr{B}(\overline{\mathbb{R}})$-measurable. By the transformation theorem, Fubini's Theorem, and the independence of X^-, V, we have

$$\begin{aligned}
\int_{\mathbb{R}_+} \mathrm{E}\left(X_t^-\right) P^V(\mathrm{d}t) &= \int_{\mathbb{R}_+} \mathrm{E}\left(g(X^-, t)\right) P^V(\mathrm{d}t) \\
&= \int_{\mathbb{R}_+} \int g(X^-, t)\, \mathrm{d}P\, P^V(\mathrm{d}t) \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{D}(\mathbb{R})} g(x,t)\, P^{X^-}(\mathrm{d}x)\, P^V(\mathrm{d}t) \\
&= \int_{\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+} g(x,t) \left(P^{X^-} \otimes P^V\right)(\mathrm{d}(x,t)) \\
&= \int_{\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+} g(x,t)\, P^{(X^-, V)}(\mathrm{d}(x,t)) \\
&= \int g(X^-, V)\, \mathrm{d}P = \mathrm{E}\left(X_V^-\right) = 0.
\end{aligned}$$

Hence, $t \mapsto \mathrm{E}\left(X_V^-\right)$ vanishes P^V-almost everywhere. Setting $h = 0$ on the P^V-null set where this is not the case, we have $h(V) \geq 0$. (For the remainder of the proof, we will omit this pathological null set.) Moreover, $h(V)$ is $\mathscr{G}$-measurable as composition of the measurable function h and the $\mathscr{G}$-measurable random variable V. Now, let $G \in \mathscr{G}$, and set $Z := 1_G$. Then, we have by Fubini's Theorem and by the independence of (Z, V) and X

$$\begin{aligned}
\int 1_G h(V) \mathrm{d}P &= \int Z \int_{\mathbb{D}(\mathbb{R})} g(x,V) P^X(\mathrm{d}x) \mathrm{d}P \\
&= \int_{\mathbb{R}_+^2} z \int_{\mathbb{D}(\mathbb{R})} g(x,v) P^X(\mathrm{d}x) P^{(Z,V)}(\mathrm{d}(z,v)) \\
&= \int_{\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+^2} z g(x,v) \left(P^X \otimes P^{(Z,V)}\right)(\mathrm{d}(x,z,v)) \\
&= \int_{\mathbb{D}(\mathbb{R}) \times \mathbb{R}_+^2} z g(x,v) \left(P^{(X,Z,V)}\right)(\mathrm{d}(x,z,v)) \\
&= \int Z g(X,V) \mathrm{d}P \\
&= \int 1_G X_V \mathrm{d}P.
\end{aligned}$$

By definition of the conditional expectation, this shows $\mathrm{E}\left(X_V \,|\, \mathscr{G}\right) = h(V)$.

3. Now, we consider the case that X_V is integrable. Let us first prove that h is well defined. Assume that there exists a measurable set $A \subset \mathbb{R}_+$ with $P^V(A) > 0$ such that $\mathrm{E}\left(X_t^-\right) = \mathrm{E}\left(X_t^+\right) = \infty$ on A. We can then conclude as in 2. that

$$\infty = \int_A \mathrm{E}\left(X_t^-\right) P^V(\mathrm{d}t) \leq \int_{\mathbb{R}_+} \mathrm{E}\left(X_t^-\right) P^V(\mathrm{d}t) = \mathrm{E}\left(X_V^-\right),$$

and analogously for X^+, which yields a contradiction to the assumption that X_V is integrable. The integrability of $h(V)$ follows from the second calculation in 2. by setting $G = \Omega$ and by considering $|h|$ resp. $|g|$ (after applying the triangle inequality) instead of h and g. Thus Fubini's Theorem is indeed applicable, and we may conclude as in 2., which completes the proof.

□

Lemma A.0.5. *On a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$, let M be a local martingale that is exponentially special (cf. Definition E.0.10) and that allows for differential characteristics (b, c, F) with respect to a truncation function $h : \mathbb{R} \to \mathbb{R}$. Then, the exponential compensator $K(M)$ of M (cf. Definition E.0.12) is non-negative up to an evanescent set.*

PROOF. By Proposition E.0.14,

$$K(M)_t = \int_0^t \left(b_s + \frac{1}{2} c_s + \int e^x - 1 - h(x) F_s(\mathrm{d}x) \right) \mathrm{d}s, \quad t \in \mathbb{R}_+.$$

By a Taylor expansion with integral remainder term, we obtain

$$e^x - 1 - h(x) = x - h(x) + x^2 \int_0^1 e^{ux}(1-u)\,\mathrm{d}u, \quad x \in \mathbb{R}.$$

Since M is a local martingale, we have $\int_0^t \int_{\{|x|\geq 1\}} |x| \, F_s(\mathrm{d}x)\,\mathrm{d}s < \infty$ for all $t \in \mathbb{R}_+$, and

$$b + \int (x - h(x)) F(\mathrm{d}x) = 0$$

$(P \otimes \lambda)$-almost everywhere by Proposition E.0.8. By these equations and the fact that $c \geq 0$, the assertion follows from the representation of $K(M)$. □

Lemma A.0.6. *On a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$, consider a stopping time T and a special semimartingale X with decomposition*

$$X = X_0 + M + A$$

with $M \in \mathscr{M}_{\mathrm{loc}}$ and $A \in \mathscr{V}$ and predictable. Then, also the stopped process X^T is a special semimartingale with decomposition

$$X^T = X_0 + M^T + A^T,$$

where $M^T \in \mathscr{M}_{\mathrm{loc}}$ and $A^T \in \mathscr{V}$ and predictable.

PROOF. Obviously, we have the representation $X^T = X_0 + M^T + A^T$, and $M^T \in \mathscr{M}_{\mathrm{loc}}$ since $M \in \mathscr{M}_{\mathrm{loc}}$. Moreover, $A^T \in \mathscr{V}$ and predictable by [JS03, Proposition I.2.4] since A is predictable. □

Lemma A.0.7. *On a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$, consider two semimartingales X and Y and a stopping time T. Then,*

$$\left[X^T, Y^T\right] = [X,Y]^T.$$

PROOF. By definition of $[\cdot,\cdot]$ and [JS03, I.4.37] applied twice,

$$\begin{aligned}\left[X^T, Y^T\right] &= X^T Y^T - X_0 Y_0 + \left(X^T\right)_- \cdot Y^T - \left(Y^T\right)_- \cdot X^T \\ &= X^T Y^T - X_0 Y_0 - \left(\left(X^T\right)_- 1_{[\![0,T]\!]}\right) \cdot Y - \left(\left(Y^T\right)_- 1_{[\![0,T]\!]}\right) \cdot X \\ &= X^T Y^T - X_0 Y_0 - \left(X_- 1_{[\![0,T]\!]}\right) \cdot Y - \left(Y_- 1_{[\![0,T]\!]}\right) \cdot X \\ &= X^T Y^T - X_0 Y_0 - (X_- \cdot Y)^T - (Y_- \cdot X)^T \\ &= (XY - X_0 Y_0 - X_- \cdot Y - Y_- \cdot X)^T \\ &= [X,Y]^T. \end{aligned}$$

□

Lemma A.0.8. *On a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$, consider a stopping time T and two semimartingales X and Y such that $[X,Y]$ is a special semimartingale. Then, also $\left[X^T, Y^T\right]$ is special, and*

$$\langle X^T, Y^T \rangle = \langle X, Y \rangle^T.$$

PROOF. By Lemma A.0.7, $\left[X^T, Y^T\right] = [X,Y]^T$, and hence $\left[X^T, Y^T\right]$ is special by Lemma A.0.6 as a stopped special semimartingale. On one hand, $\langle X, Y\rangle$ is by definition the unique compensator of $[X,Y]$, and by Lemma A.0.6, $\langle X, Y\rangle^T$ is thus the unique compensator of $[X,Y]^T$. On the other hand, $\langle X^T, Y^T\rangle$ is the unique compensator of $\left[X^T, Y^T\right] = [X,Y]^T$. Hence, $\langle X^T, Y^T\rangle = \langle X, Y\rangle^T$. □

B. Moments of the integrated square root process and the integrated squared Gaussian Ornstein-Uhlenbeck process

In this chapter, we derive explicit representations for certain moments of the integrated square root process and the integrated squared Gaussian Ornstein-Uhlenbeck (OU) process. These moments are required for the approximation to option prices in Theorem 5.4.3 in the Heston, NIG-CIR, and the Stein & Stein model.

B.1. Square root process

The square root process used in the models of Section 5.5.2.1 and Section 5.5.5.1 is given by

$$\mathrm{d}y_t = \kappa(\eta - y_t)\,\mathrm{d}t + \theta\sqrt{y_t}\,\mathrm{d}W_t, \quad y_0 > 0, \tag{B.1}$$

for a standard Brownian motion W^1, κ, η, $\theta > 0$. By [BJ08, Proposition 4.1, A.15], the process y has mean and covariance function given by

$$\begin{aligned}
\mathrm{E}\left(y_t\right) &= e^{-\kappa t}\left(y_0 - \eta\right) + \eta, \\
\mathrm{Cov}\left(y_s, y_t\right) &= \theta^2 e^{-\kappa(s+t)}\left(\frac{e^{\kappa\min\{s,t\}} - 1}{\kappa}\left(y_0 - \eta\right) + \eta\,\frac{e^{2\kappa\min\{s,t\}} - 1}{2\kappa}\right),
\end{aligned}$$

$s,t \in \mathbb{R}_+$. For $V_T := \int_0^T y_t\,\mathrm{d}t$, $T \in \mathbb{R}_+$, we have by Fubini's Theorem

$$\begin{aligned}
\mathrm{E}\left(V_T\right) &= \int_0^T \mathrm{E}\left(y_t\right)\,\mathrm{d}t \\
&= \eta T + \frac{(y_0 - \eta)(1 - e^{-\kappa T})}{\kappa}, \\
\mathrm{Var}\left(V_T\right) &= \int_0^T \int_0^T \mathrm{Cov}\left(y_s, y_t\right)\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{\theta^2}{2\kappa^3}\Big(2y_0 - 5\eta - 2y_0 e^{-2\kappa T} - 4y_0 e^{-\kappa T}\kappa T \\
&\qquad + \eta e^{-2\kappa T} + 4\eta e^{-\kappa T}\left(\kappa T + 1\right) + 2\eta T\kappa\Big).
\end{aligned}$$

In order to compute $\mathrm{Cov}(V_T, M_T)$ for $M_T := \int_0^T \sqrt{y_t}\,\mathrm{d}W_t$, $T \in \mathbb{R}_+$, it is useful to observe that by (B.1),

$$M_T = \frac{1}{\theta}\left(y_T - y_0 - \int_0^T \kappa(\eta - y_t)\,\mathrm{d}t\right),$$

and hence

$$\begin{aligned}\mathrm{Cov}(V_T, M_T) &= \frac{1}{\theta}\left(\mathrm{Cov}(V_T, y_T) + \kappa \mathrm{Cov}\left(V_T, \int_0^T y_t\,\mathrm{d}t\right)\right)\\ &= \frac{1}{\theta}\left(\int_0^T \mathrm{Cov}(y_t, y_T)\,\mathrm{d}t + \kappa \mathrm{Var}(V_T)\right),\end{aligned}$$

where $\mathrm{Var}(V_T)$ has already been computed above. The time integral can be evaluated explicitly:

$$\int_0^T \mathrm{Cov}(y_t, y_T)\,\mathrm{d}t = \frac{\theta^2}{2\kappa^2}\left(2e^{-\kappa T}(y_0\kappa T - y_0 - \eta\kappa T) + e^{-2\kappa T}(2y_0 - \eta) + \eta\right).$$

B.2. Gaussian Ornstein-Uhlenbeck process

B.2.1. Squared normally distributed random variables

Lemma B.2.1. *Let $(X,Y) \sim N(\mu, \Sigma)$ be a bivariate normally distributed random vector with mean vector μ and covariance matrix Σ. Then, we have*

$$\begin{aligned}\mathrm{Cov}\left(X^2, Y\right) &= 2\mathrm{Cov}(X,Y)\,\mathrm{E}(X),\\ \mathrm{Cov}\left(X^2, Y^2\right) &= 4\mathrm{E}(X)\,\mathrm{E}(Y)\,\mathrm{Cov}(X,Y) + 2\mathrm{Cov}(X,Y)^2.\end{aligned}$$

Proof. We write $\mu = \begin{pmatrix}\mu_X\\ \mu_Y\end{pmatrix}$ and $\Sigma = \begin{pmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y\\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{pmatrix}$. It is a standard result that

$$Y|X = x \sim N\left(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x - \mu_X), (1-\rho^2)\sigma_Y^2\right).$$

Hence,

$$\begin{aligned}\mathrm{E}\left(X^2 Y\right) &= \mathrm{E}\left(X^2 \mathrm{E}(Y|X)\right)\\ &= \mathrm{E}\left(X^2\left(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(X - \mu_X)\right)\right)\\ &= \mu_Y\left(\mu_X^2 + \sigma_X^2\right) + 2\rho\sigma_X\sigma_Y\mu_X.\end{aligned}$$

Since $\mathrm{E}\left(X^2\right)\mathrm{E}(Y) = \mu_Y\left(\mu_X^2 + \sigma_X^2\right)$, we obtain

$$\mathrm{Cov}\left(X^2, Y\right) = 2\rho\sigma_X\sigma_Y\mu_X = 2\mathrm{Cov}(X,Y)\,\mathrm{E}(X).$$

Analogously, we derive

$$\begin{aligned}\mathrm{Cov}\left(X^2, Y^2\right) &= 4\mu_X\mu_Y\rho\sigma_X\sigma_Y + 2\rho^2\sigma_X^2\sigma_Y^2\\ &= 4\mathrm{E}(X)\,\mathrm{E}(Y)\,\mathrm{Cov}(X,Y) + 2\mathrm{Cov}(X,Y)^2.\end{aligned}$$

□

B.2.2. Moments of the integrated squared Gaussian OU process

The Gaussian Ornstein-Uhlenbeck process used in the model from Section 5.5.3.1 is given by

$$\mathrm{d}y_t = \kappa(\eta - y_t)\,\mathrm{d}t + \theta\,\mathrm{d}W_t, \quad y_0 > 0,$$

for a standard Brownian motion W, κ, η, $\theta > 0$. It is well known that y is a Gaussian process with mean and covariance function

$$\begin{aligned}\mathrm{E}\left(y_t\right) &= y_0 e^{-\kappa t} + \eta\left(1 - e^{-\kappa t}\right),\\ \mathrm{Cov}\left(y_s, y_t\right) &= \frac{\theta^2}{2\kappa} e^{-\kappa(s+t)}\left(e^{2\kappa\min\{s,t\}} - 1\right)\end{aligned}$$

for $s,t \in \mathbb{R}_+$. Hence, we have

$$\mathrm{E}\left(y_t^2\right) = \left(y_0 e^{-\kappa t} + \eta\left(1 - e^{-\kappa t}\right)\right)^2 + \frac{\theta^2}{2\kappa}\left(1 - e^{-2\kappa t}\right),$$

and Lemma B.2.1 yields

$$\begin{aligned}\mathrm{Cov}\left(y_s^2, y_t^2\right) = \frac{2\theta^2}{\kappa}&\left(y_0 e^{-\kappa s} + \eta\left(1 - e^{-\kappa s}\right)\right)\left(y_0 e^{-\kappa t} + \eta\left(1 - e^{-\kappa t}\right)\right) &\text{(B.2)}\\ &\cdot e^{-\kappa(s+t)}\left(e^{2\kappa\min\{s,t\}} - 1\right) + \frac{\theta^4}{2\kappa^2} e^{-2\kappa(s+t)}\left(e^{2\kappa\min\{s,t\}} - 1\right)^2. &\text{(B.3)}\end{aligned}$$

Setting $V_T := \int_0^T y_t^2\,\mathrm{d}t$, $T \in \mathbb{R}_+$, we have by Fubini's Theorem

$$\begin{aligned}\mathrm{E}\left(V_T\right) &= \int_0^T \mathrm{E}\left(y_t^2\right)\,\mathrm{d}t,\\ \mathrm{Var}\left(V_T\right) &= \int_0^T\int_0^T \mathrm{Cov}\left(y_s^2, y_t^2\right)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Hence, the computation of $\mathrm{E}\left(V_T\right)$ and $\mathrm{Var}\left(V_T\right)$ amounts to the evaluation of a single and double integral over essentially exponential functions. Delegating this tedious work to the symbolic software package *Maple* yields

$$\begin{aligned}\mathrm{E}\left(V_T\right) = -\frac{1}{4\kappa^2}\Big(&\theta^2 - 4y_0\eta\kappa - 2{y_0}^2\kappa + 6\kappa\eta^2 - 2\theta^2 T\kappa - \theta^2 e^{-2\kappa T}\\ &- 4e^{-2\kappa T} y_0\eta\kappa + 2e^{-2\kappa T}{y_0}^2\kappa + 2e^{-2\kappa T}\eta^2\kappa - 8\eta^2 e^{-\kappa T}\kappa\\ &+ 8y_0 e^{-\kappa T}\eta\kappa - 4\eta^2\kappa^2 T\Big),\\ \mathrm{Var}\left(V_T\right) = \frac{\theta^2}{8\kappa^4} e^{-4\kappa T}\Big(&-5e^{4\kappa T}\theta^2 + 24e^{4\kappa T} y_0\eta\kappa - 76e^{4\kappa T}\kappa\eta^2 + 4e^{4\kappa T}{y_0}^2\kappa\\ &+ 4\theta^2 e^{2\kappa T} + 32y_0\eta e^{2\kappa T}\kappa - 48y_0\eta e^{3\kappa T}\kappa - 16y_0\eta e^{\kappa T}\kappa - 32y_0\eta e^{3\kappa T}\kappa^2 T\\ &+ 32y_0\eta e^{2\kappa T}\kappa^2 T - 4\kappa\eta^2 + 32\kappa^2\eta^2 T e^{4\kappa T} + 4\theta^2 T e^{4\kappa T}\kappa + 112\eta^2 e^{3\kappa T}\kappa\\ &- 48\eta^2 e^{2\kappa T}\kappa + 16\eta^2 e^{\kappa T}\kappa + 32\eta^2 e^{3\kappa T}\kappa^2 T + 8\theta^2 e^{2\kappa T}\kappa T - 16{y_0}^2 e^{2\kappa T}\kappa^2 T\\ &- 16\eta^2 e^{2\kappa T}\kappa^2 T + \theta^2 - 4{y_0}^2\kappa + 8y_0\eta\kappa\Big).\end{aligned}$$

In order to compute $\mathrm{Cov}(V_T, M_T)$ with $M_T := \int_0^T y_t \, \mathrm{d}W_t$, $T \in \mathbb{R}_+$, let us first observe that by Itō's Lemma, we obtain

$$\mathrm{d}y_t^2 = \left(2\kappa y_t (\eta - y_t) + \theta^2\right) \mathrm{d}t + 2\theta y_t \, \mathrm{d}W_t.$$

Hence,

$$M_T = \frac{1}{2\theta} \left(y_T^2 - y_0^2 - \int_0^T \left(2\kappa y_t (\eta - y_t) + \theta^2\right) \mathrm{d}t \right).$$

Therefore, we have

$$\begin{aligned}
\mathrm{Cov}(V_T, M_T) &= \frac{1}{2\theta} \mathrm{Cov}\left(\int_0^T y_t^2 \, \mathrm{d}t, y_T^2 - 2\kappa\eta \int_0^T y_t \, \mathrm{d}t + 2\kappa \int_0^T y_t^2 \, \mathrm{d}t \right) \\
&= \frac{1}{2\theta} \left(2\kappa \mathrm{Var}(V_T) + \int_0^T \mathrm{Cov}\left(y_T^2, y_t^2\right) \mathrm{d}t - 2\kappa\eta \int_0^T \int_0^T \mathrm{Cov}\left(y_s, y_t^2\right) \mathrm{d}s \, \mathrm{d}t \right).
\end{aligned}$$

By (B.2), we obtain

$$\begin{aligned}
\int_0^T \mathrm{Cov}\left(y_T^2, y_t^2\right) \mathrm{d}t = \frac{\theta^2}{4\kappa^3} e^{-4\kappa T} \Big(&-20\eta^2 e^{3\kappa T}\kappa - 4\kappa e^{2\kappa T} {y_0}^2 - 8y_0\eta e^{2\kappa T}\kappa \\
&+ 20\eta^2 e^{2\kappa T}\kappa + 4y_0\eta e^{3\kappa T}\kappa + 12y_0\eta e^{\kappa T}\kappa - 8y_0\eta\kappa \\
&- 16y_0\eta e^{2\kappa T}\kappa^2 T + 8y_0\eta e^{3\kappa T}\kappa^2 T - 12\eta^2 e^{\kappa T}\kappa \\
&+ 8\eta^2 e^{2\kappa T}\kappa^2 T + 4\kappa\eta^2 + e^{4\kappa T}\theta^2 + 8e^{4\kappa T}\kappa\eta^2 \\
&+ 4{y_0}^2\kappa - \theta^2 + 8{y_0}^2 e^{2\kappa T}\kappa^2 T - 4\theta^2 e^{2\kappa T}\kappa T - 8\eta^2 e^{3\kappa T}\kappa^2 T \Big).
\end{aligned}$$

We conclude from Lemma B.2.1 that

$$\mathrm{Cov}\left(y_s, y_t^2\right) = \frac{\theta^2}{\kappa} e^{-\kappa(s+t)} \left(e^{2\kappa \min\{s,t\}} - 1 \right) \left(y_0 e^{-\kappa t} + \eta\left(1 - e^{-\kappa t}\right)\right),$$

and we obtain

$$\begin{aligned}
\int_0^T \int_0^T \mathrm{Cov}\left(y_s, y_t^2\right) \mathrm{d}s\mathrm{d}t = -\frac{\theta^2}{2\kappa^3} e^{-3\kappa T} \Big(&-2e^{3\kappa T} y_0 + 8\eta e^{3\kappa T} - 11\eta e^{2\kappa T} \\
&+ 3y_0 e^{2\kappa T} - 2y_0 e^{\kappa T} + 4\eta e^{\kappa T} + 2y_0 T\kappa e^{2\kappa T} + y_0 \\
&- \eta - 4\eta T\kappa e^{3\kappa T} - 2\eta T\kappa e^{2\kappa T} \Big).
\end{aligned}$$

C. Interchanging differentiation and integration

Even if the following theorem on the interchange of integration and differentiation of a parameter integral is contained every textbook on measure theory, we recapitulate it here since we use it over and over again in this thesis.

Theorem C.0.2. *Let $(X, \mathscr{A}, \mu)$ be a measure space. Moreover, consider an interval $I \subset \mathbb{R}$, $t_0 \in I$, and a function $f : I \times X \to \mathbb{C}, (t,x) \mapsto f(t,x)$, with the following properties:*

1. *For all $t \in I$, we have $(x \mapsto f(t,x)) \in L^1(X, \mathscr{A}, \mu)$.*
2. *There exists $\delta > 0$ such that for all $x \in X$ and all $t \in U := (t_0 - \delta, t_0 + \delta) \cap I$, the partial (eventually one-sided) derivative $\frac{\partial f}{\partial t}(t,x)$ exists.*
3. *There exists a non-negative function $g \in L^1(X, \mathscr{A}, \mu)$ such that for all $t \in U$ and all $x \in X$,*
$$\left| \frac{\partial f}{\partial t}(t,x) \right| \leq g(x).$$

Then, the function
$$F : I \to \mathbb{C}, \quad t \mapsto \int_X f(t,x) \, \mathrm{d}\mu(x)$$
is (eventually one-sided) differentiable in t_0, the mapping $\left(x \mapsto \frac{\partial f}{\partial t}(t_0,x)\right) \in L^1(X, \mathscr{A}, \mu)$, and we have
$$F'(t_0) = \int_X \frac{\partial f}{\partial t}(t_0,x) \, \mathrm{d}\mu(x).$$

Proof. Cf. [Els05, Satz 5.7]. □

In this work, we are usually in the situation that partial differentiability is given on the whole interval of interest, and we are typically interested also in higher-order derivatives. This situation is treated in the following

Corollary C.0.3. *Let $(X, \mathscr{A}, \mu)$ be a measure space. Moreover, consider an interval $I \subset \mathbb{R}$, $n \in \mathbb{N}_{\geq 1}$, and a function $f : I \times X \to \mathbb{C}, (t,x) \mapsto f(t,x)$, with the following properties:*

1. *For all $t \in I$, we have $(x \mapsto f(t,x)) \in L^1(X, \mathscr{A}, \mu)$.*

2. *For all $t \in I$, all $x \in X$, and all $k \in \{1, \dots, n\}$, the (eventually one-sided) partial derivative* $\frac{\partial^k f}{\partial t^k}(t,x)$ *exists.*
3. *There exists a non-negative function $g \in L^1(X, \mathscr{A}, \mu)$ such that for all $t \in I$, all $k \in \{1, \dots, n\}$, and all $x \in X$,*
$$\left| \frac{\partial^k f}{\partial t^k}(t,x) \right| \leq g(x).$$

Then, the function

$$F : I \to \mathbb{C}, \quad t \mapsto \int_X f(t,x) \, \mathrm{d}\mu(x),$$

is (eventually one-sided) n-times differentiable, and we have

$$F^{(k)}(t) = \int_X \frac{\partial^k f}{\partial t^k}(t,x) \, \mathrm{d}\mu(x), \quad k \in \{1, \dots, n\}, \quad t \in I.$$

Proof. This follows directly by iterated application of Theorem C.0.2. □

D. Sensitivities of Black-Scholes call and put prices

On some suitable probability space, consider the risk-neutral price process of the stock in the Black-Scholes model with interest rate $r \geq 0$ and volatility parameter $\sigma > 0$, given by

$$dS_t = rS_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \quad S_0 \in \mathbb{R}_+,$$

for standard Brownian motion W. Let C be the pricing function of a European call with strike $K > 0$ and maturity $T > 0$ in this model, i.e., $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ with

$$C(t,s) = e^{-r(T-t)} \mathrm{E}\left((S_T - K)^+ \middle| S_t = s \right).$$

For all $t \in [0,T)$, this function is infinitely differentiable with respect to s (cf. Lemma 3.4.1), and we set for $n \in \mathbb{N}$ and $t \in [0,T)$

$$d_n(t,s) := \frac{\partial^n C}{\partial s^n}(t,s).$$

With Φ and φ being the cumulative distribution function and the density of the standard normal distribution, respectively, we obtain from tedious but straightforward computations for $s > 0$

$$\begin{aligned}
d_0(t,s) &= s\Phi(\delta_1(t,s)) - e^{-r(T-t)} K\Phi(\delta_2(t,s)), \\
d_1(t,s) &= \Phi(\delta_1(t,s)), \\
d_2(t,s) &= \frac{\varphi(\delta_1(t,s))}{s\sigma\sqrt{(T-t)}}, \\
d_3(t,s) &= -\frac{d_2(t,s)}{s}\left(\frac{\delta_1(t,s)}{\sigma\sqrt{T-t}} + 1\right), \\
d_4(t,s) &= \frac{d_2(t,s)}{s^2}\left(\frac{\delta_1(t,s)}{\sigma\sqrt{T-t}} + 1 - \frac{1}{\sigma^2(T-t)}\right) - \frac{d_3(t,s)}{s}\left(\frac{\delta_1(t,s)}{\sigma\sqrt{T-t}} + 1\right), \\
d_5(t,s) &= \frac{d_2(t,s)}{s^3}\left(\frac{3}{\sigma^2(T-t)} - \frac{2\delta_1(t,s)}{\sigma\sqrt{T-t}} - 2\right) \\
&\quad + \frac{2d_3(t,s)}{s^2}\left(\frac{\delta_1(t,s)}{\sigma\sqrt{T-t}} + 1 - \frac{1}{\sigma^2(T-t)}\right) \\
&\quad - \frac{d_4(t,s)}{s}\left(\frac{\delta_1(t,s)}{\sigma\sqrt{T-t}} + 1\right), \\
d_6(t,s) &= \frac{d_2(t,s)}{s^4}\left(\frac{6\delta_1(t,s)}{\sigma\sqrt{T-t}} - \frac{9}{\sigma^2(T-t)} + 6 - \frac{2}{\sigma^2(T-t)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{d_3(t,s)}{s^3}\left(\frac{9}{\sigma^2(T-t)}-\frac{6\delta_1(t,s)}{\sigma\sqrt{T-t}}-6\right)\\
&+\frac{d_4(t,s)}{s^2}\left(\frac{3\delta_1(t,s)}{\sigma\sqrt{T-t}}-\frac{3}{\sigma^2(T-t)}+3\right)-\frac{d_5(t,s)}{s}\left(\frac{\delta_1(t,s)}{\sigma\sqrt{T-t}}+1\right)
\end{aligned}
$$

for

$$
\begin{aligned}
\delta_1(t,s) &= \frac{\log\left(\frac{s}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},\\
\delta_2(t,s) &= \delta_1(t,s)-\sigma\sqrt{T-t}.
\end{aligned}
$$

Here, $d_0(t,s) = C(t,s)$ is the famous Black-Scholes formula for the call option price.

If we consider a European put with strike $K > 0$ and maturity $T > 0$ instead, the formulas for $d_2(t,s),\dots,d_6(t,s)$ are identical. For the price and the first derivative, we have

$$
\begin{aligned}
d_0(t,s) &= \Phi(-\delta_2(t,s))Ke^{-r(T-t)} - \Phi(-\delta_1(t,s))s,\\
d_1(t,s) &= \Phi(\delta_1(t,s)) - 1.
\end{aligned}
$$

E. Differential characteristics and exponential compensators

In this appendix, we present the definition of differential characteristics and several related statements that we use in this work. For an excellent motivation and introduction to differential semimartingale calculus, we refer the reader to [Kal06].

Throughout, we work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$.

Definition E.0.4. [Differential characteristics] Let X be an $\mathbb{R}^d$-valued semimartingale with integral characteristics (B, C, ν) relative to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ in the sense of [JS03, Definition II.2.6]. By [JS03, Proposition II.2.9], there exist a predictable $\mathbb{R}^d$-valued process b, a predictable process c with values in the real, non-negative definite symmetric $d\times d$-matrices, a transition kernel F from $(\Omega\times\mathbb{R}_+, \mathscr{P})$ to $(\mathbb{R}^d, \mathscr{B}^d)$, and a predictable process $A \in \mathscr{A}^+_{\mathrm{loc}}$ such that for all $t \in \mathbb{R}_+$

$$\begin{aligned}
B_t &= \int_0^t b_s \,\mathrm{d}A_s, \\
C_t &= \int_0^t c_s \,\mathrm{d}A_s, \\
\nu([0,t]\times G) &= \int_0^t F_s(G)\,\mathrm{d}A_s \quad \text{for all } G \in \mathscr{B}^d.
\end{aligned}$$

If one can choose $A = I$, we call the corresponding triplet (b, c, F) *differential characteristics* of X (relative to the truncation function h).

Remark E.0.5. For an $\mathbb{R}^d$-valued Lévy process X with Lévy-Khintchine triplet (b, c, F) relative to a truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$, the triplet (b, c, F) coincides with the differential characteristics of X relative to h. Hence, differential characteristics can be seen as “local Lévy-Khintchine triplets” of a semimartingale. Moreover, a semimartingale with constant and deterministic differential characteristics (b, c, F) is a Lévy process with Lévy-Khintchine triplet (b, c, F), cf. [JS03, Corollary II.4.19].

Proposition E.0.6 (Itō’s formula for differential characteristics)**.** *Let X be an $\mathbb{R}^d$-valued semimartingale with differential characteristics (b, c, F) relative to a truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. Moreover, consider a twice continuously differentiable function $f : U \to \mathbb{R}^n$ on an open subset*

$U \subseteq \mathbb{R}^d$ such that X and X_- are U-valued. Then, the triplet $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ given by

$$\begin{aligned}
\widetilde{b}_t &= Df(X_{t-})b_t + \frac{1}{2}\sum_{j,k=1}^{d} D_{jk}f(X_{t-})c_t^{jk} \\
&\quad + \int \Big(\widetilde{h}(f(X_{t-}+x)-f(X_{t-})) - Df(X_{t-})h(x)\Big)\, F_t(\mathrm{d}x), \\
\widetilde{c}_t &= Df(X_{t-})c_t(Df(X_{t-}))^\top, \\
\widetilde{F}_t(G) &= \int 1_G(f(X_{t-}+x)-f(X_{t-}))\, F_t(\mathrm{d}x) \quad \text{for all } G \in \mathscr{B}^n \text{ with } 0 \notin G
\end{aligned}$$

is a version of the differential characteristics of $f(X)$ with respect to an arbitrary truncation function $\widetilde{h}$ on $\mathbb{R}^n$. Here, Df denotes the Jacobian of f, and D_{jk} denote partial derivatives of f with respect to the i-th and j-th component.

PROOF. Cf. [GK00, Corollary A.6]. □

Remark E.0.7. We remark that there exist also rules how differential characteristics of a semimartingale transform under stochastic integration (cf. [KS02b, Lemma 3]), change of measure (cf. [Kal04, Lemma 5.1]), and absolutely continuous time change (cf. [Kal06, Proposition 2.7]). Since we dot not need these tools in this work, we do not repeat the corresponding statements but present only the references.

We come now to some properties of a semimartingale that can be read from its differential characteristics, and we present representations of the predictable covariation and the exponential compensator in terms of the differential characteristics.

Proposition E.0.8 (Martingale property and differential characteristics). *Let X be a real-valued semimartingale with differential characteristics (b,c,F) relative to a truncation function $h : \mathbb{R} \to \mathbb{R}$.*

1. *X is a local martingale if and only if*

$$b + \int (x - h(x))\, F(\mathrm{d}x) = 0 \tag{E.1}$$

   holds $P \otimes \lambda$-everywhere, and $\int_0^t \int_{\{|x|\geq 1\}} |x|\, F_s(\mathrm{d}x)\,\mathrm{d}t < \infty$ for all $t \in \mathbb{R}_+$, where λ denotes the Lebesgue measure.

2. *X is a martingale if and only if* (E.1) *holds $P \otimes \lambda$-everywhere and for all $t \in \mathbb{R}_+$, the stopped process X^t is of class (D) in the sense of [JS03, Definition I.1.46].*

PROOF. Cf. [Kal04, Lemma 3.1]. □

Proposition E.0.9 (Predictable covariation and differential characteristics). *Let X be an $\mathbb{R}^d$-valued semimartingale with differential characteristics (b,c,F) relative to a truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. Assume further that $[X,X]$ is special. Then, we have for the predictable covariation of X*

$$\langle X, X\rangle_t = \int_0^t c_s\,\mathrm{d}s + \int_0^t \int xx^\top\, F_s(\mathrm{d}x)\,\mathrm{d}s, \quad t \in \mathbb{R}_+.$$

PROOF. Cf. [ČernýK07, Proposition 1.2]. □

Definition E.0.10. Let X be a real-valued semimartingale. X is called *exponentially special* if $\exp(X - X_0)$ is a special semimartingale.

Proposition E.0.11 (Exponentially special semimartingales and differential characteristics)**.** *Let X be a real-valued semimartingale with differential characteristics (b, c, F) relative to a truncation function $h : \mathbb{R} \to \mathbb{R}$. Then the following statements are equivalent.*

1. *X is exponentially special.*
2. *$\int_0^t \int (e^x - 1 - h(x))\, F_s(\mathrm{d}x)\, \mathrm{d}s < \infty$ for all $t \in \mathbb{R}_+$.*
3. *$\int_0^t \int_{\{x>1\}} e^x F_s(\mathrm{d}x)\, \mathrm{d}s < \infty$ for all $t \in \mathbb{R}_+$.*

PROOF. Cf. [KS02a, Lemma 2.13]. □

Definition E.0.12. Let X be a real-valued semimartingale. A predictable process $V \in \mathscr{V}$ is called *exponential compensator* of X if $\exp(X - X_0 - V) \in \mathscr{M}_{\mathrm{loc}}$.

Lemma E.0.13. *A real-valued semimartingale X has an exponential compensator if and only if it is exponentially special. In this case, the exponential compensator is unique up to indistinguishability.*

PROOF. Cf. [KS02a, Lemma 2.15]. □

Proposition E.0.14 (Exponential compensator and differential characteristics)**.** *Let X be an $\mathbb{R}^d$-valued semimartingale with differential characteristics (b, c, F) relative to a truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. Moreover, consider an $\mathbb{R}^d$-valued process $\vartheta \in L(X)$ such that $\int_0^\cdot \vartheta_t \, \mathrm{d}X_t$ is exponentially special. Then, the exponential compensator $K(\int_0^\cdot \vartheta_t \, \mathrm{d}X_t)$ of $\int_0^\cdot \vartheta_t \, \mathrm{d}X_t$ is given by*

$$K\left(\int_0^\cdot \vartheta_t \, \mathrm{d}X_t\right)_t = \int_0^t \left(\vartheta_s^\top b_s + \frac{1}{2}\vartheta_s^\top c_s \vartheta_s + \int \left(e^{\vartheta_s^\top x} - 1 - \vartheta_s^\top h(x)\right) F_s(\mathrm{d}x)\right) \mathrm{d}s, \quad t \in \mathbb{R}_+.$$

PROOF. This follows from [KS02a, Theorem 2.18(1,2) and Theorem 2.19] . □

Bibliography

[Alò06] E. Alòs. A generalization of the Hull and White formula with applications to option pricing approximation. *Finance and Stochastics*, 10(3):353–365, 2006.

[Alò12] E. Alòs. A decomposition formula for option prices in the Heston model and applications to option pricing approximation. *Finance and Stochastics*, 16(3):403–422, 2012.

[AP07] L. Andersen and V. Piterbarg. Moment explosions in stochastic volatility models. *Finance and Stochastics*, 11(1):29–50, 2007.

[AS09] F. Antonelli and S. Scarlatti. Pricing options under stochastic volatility: a power series approach. *Finance and Stochastics*, 13(2):269–303, 2009.

[AW95] C. Atkinson and P. Wilmott. Portfolio management with transaction costs: an asymptotic analysis of the Morton and Pliska model. *Mathematical Finance*, 5(4):357–367, 1995.

[Bau78] H. Bauer. *Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie*. de Gruyter, 3. edition, 1978.

[BGM09] E. Benhamou, E. Gobet, and M. Miri. Smart expansion and fast calibration for jump diffusions. *Finance and Stochastics*, 13(4):563–589, 2009.

[BGM10a] E. Benhamou, E. Gobet, and M. Miri. Expansion formulas for European options in a local volatility model. *International Journal of Theoretical and Applied Finance*, 13(4):603–634, 2010.

[BGM10b] E. Benhamou, E. Gobet, and M. Miri. Time dependent Heston model. *SIAM Journal on Financial Mathematics*, 1(1):289–325, 2010.

[BJ08] M. Broadie and A. Jain. The effect of jumps and discrete sampling on volatility and variance swaps. *International Journal of Theoretical and Applied Finance*, 11(8):761–797, 2008.

[BKL00] D. Bertsimas, L. Kogan, and A. Lo. When is time continuous? *Journal of Financial Economics*, 55(2):173–204, 2000.

[BM06] D. Brigo and F. Mercurio. *Interest Rate Models – Theory and Practice*. Springer, 2. edition, 2006.

[BN95] O. Barndorff-Nielsen. *Normal inverse Gaussian processes and the modelling of stock returns*. Research report no. 300, Department of Theoretical Statistics, Institute of Mathematics, University of Aarhus, 1995.

[BN98] O. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2(1):41–68, 1998.

[BNNS02] O. Barndorff-Nielsen, E. Nicolato, and N. Shephard. Some recent developments in stochastic volatility modelling. *Quantitative Finance*, 2(1):11–23, 2002.

[BNS01] O. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society B*, 63(2):167–241, 2001.

[BNS03] O. Barndorff-Nielsen and N. Shephard. Integrated OU processes and non-Gaussian OU-based stochastic volatility models. *Scandinavian Journal of Statistics*, 30(2):277–295, 2003.

[BR94] C. Ball and A. Roma. Stochastic volatility option pricing. *Journal of Financial and Quantitative Analysis*, 29(4):589–607, 1994.

[BS73] F. Black and M. Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, 81(3):637–654, 1973.

[BS98] G. Barles and H. Soner. Option pricing with transaction costs and a nonlinear Black-Scholes equation. *Finance and Stochastics*, 2(4):369–397, 1998.

[BT11] M. Broden and P. Tankov. Tracking errors from discrete hedging in exponential Lévy models. *International Journal of Theoretical and Applied Finance*, 14(6):1–35, 2011.

[CDF02] R. Cont and J. Da Fonseca. Dynamics of implied volatility surfaces. *Quantitative Finance*, 2(1):45–60, 2002.

[Černý07] A. Černý. Optimal continuous-time hedging with leptokurtic returns. *Mathematical Finance*, 17(2):175–203, 2007.

[ČernýK07] A. Černý and J. Kallsen. On the structure of general mean-variance hedging strategies. *The Annals of Probability*, 35(4):1479–1531, 2007.

[CGMY02] P. Carr, H. Geman, D. Madan, and M. Yor. The fine structure of asset returns: an empirical investigation. *The Journal of Business*, 75(2):305–332, 2002.

[CGMY03] P. Carr, H. Geman, D. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13(3):345–382, 2003.

[CI85] J. Cox and E. Ingersoll. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.

[CM99] P. Carr and D. Madan. Option valuation using the fast Fourier transform. *The Journal of Computational Finance*, 2(4):61–73, 1999.

[CT03] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall/CRC, 2003.

[Dei05] A. Deitmar. *A First Course in Harmonic Analysis*. Springer, 2. edition, 2005.

[DFS03] D. Duffie, D. Filipovic, and W. Schachermayer. Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3):984–1053, 2003.

[DFW98] B. Dumas, J. Fleming, and R. Whaley. Implied volatility functions: empirical tests. *The Journal of Finance*, 53(6):2059–2106, 1998.

[DGMK+13] S. Denkl, M. Goy, J. Muhle-Karbe, J. Kallsen, and A. Pauwels. On the performance of delta hedging strategies in exponential Lévy models. *Quantitative Finance*, 2013. To appear.

[DN90] M. Davis and A. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713, 1990.

[Doe50] G. Doetsch. *Handbuch der Laplace-Transformation, Band I*. Birkhäuser, 1950.

[DS94] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300(1):463–520, 1994.

[EGP10] E. Eberlein, K. Glau, and A. Papapantoleon. Analysis of Fourier transform valuation formulas and applications. *Applied Mathematical Finance*, 17(3):211–240, 2010.

[EH88] J. Eastham and K. Hastings. Optimal impulse control of portfolios. *Mathematics of Operations Research*, 13(4):588–605, 1988.

[EK95] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.

[EK05] R. Elliott and P. Kopp. *Mathematics of Financial Markets*. Springer, 2. edition, 2005.

[Els05] J. Elstrodt. *Maß- und Integrationstheorie*. Springer, 4., corrected edition, 2005.

[EP00] E. Eberlein and K. Prause. The generalized hyperbolic model: financial derivatives and risk measures. In H. Geman, D. Madan, S. Pliska, and T. Vorst, editors, *Mathematical Finance – Bachelier Congress 2000*, pages 245–267, 2000.

[FL94] W. Fischer and I. Lieb. *Funktionentheorie*. Vieweg, 7. edition, 1994.

[FM09] D. Filipović and E. Mayerhofer. Affine diffusion processes: theory and applications. In *Radon Series on Computational and Applied Mathematics*, volume 8, pages 1–40, 2009.

[FPS00] J.-P. Fouque, G. Papanicolaou, and R. Sircar. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, 2000.

[FPSS03] J.-P. Fouque, G. Papanicolaou, K. Solna, and R. Sircar. Singular perturbations in option pricing. *SIAM Journal on Applied Mathematics*, 63(5):1648–1665, 2003.

[FS86] H. Föllmer and D. Sondermann. Hedging of nonredundant contingent claims. In W. Hildenbrand and A. Mas-Colell, editors, *Contributions to Mathematical Economics*, pages 205–223. North-Holland, 1986.

[Fuk11a] M. Fukasawa. Asymptotic analysis for stochastic volatility: Edgeworth expansion. *Electronic Journal of Probability*, 16(25):764–791, 2011.

[Fuk11b] M. Fukasawa. Asymptotic analysis for stochastic volatility: martingale expansion. *Finance and Stochastics*, 15(4):635–654, 2011.

[Gei02] S. Geiss. Quantitative approximation of certain stochastic integrals. *Stochastics and Stochastic Reports*, 73(3-4):241–270, 2002.

[GG06] C. Geiss and S. Geiss. On an approximation problem for stochastic integrals where random time nets do not help. *Stochastic Processes and their Applications*, 116(3):407–422, 2006.

[GHL+12] J. Gatheral, E. Hsu, P. Laurence, C. Ouyang, and T.-H. Wang. Asymptotics of implied volatility in local volatility models. *Mathematical Finance*, 22(4):591–620, 2012.

[GK00] T. Goll and J. Kallsen. Optimal portfolios for logarithmic utility. *Stochastic Processes and their Applications*, 89(1):31–48, 2000.

[Gow72] K. Gowrisankaran. Measurability of functions in product spaces. *Proceedings of the American Mathematical Society*, 31(2):485–488, 1972.

[GT01] E. Gobet and E. Temam. Discrete time hedging errors for options with irregular payoffs. *Finance and Stochastics*, 5(3):357–367, 2001.

[Hes93] S. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[HKK06] F. Hubalek, J. Kallsen, and L. Krawczyk. Variance-optimal hedging for processes with stationary independent increments. *The Annals of Applied Probability*, 16(2):853–885, 2006.

[HM05] T. Hayashi and P. Mykland. Evaluating hedging errors: an asymptotic approach. *Mathematical Finance*, 15(2):309–343, 2005.

[HW87] J. Hull and A. White. The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, 42(2):281–300, 1987.

[HW99] P. Hagan and D. Woodward. Equivalent Black volatilities. *Applied Mathematical Finance*, 6(3):147–157, 1999.

[Jac79] J. Jacod. *Calcul Stochastique et Problèmes de Martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, 1979.

[JR82] R. Jarrow and A. Rudd. Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics*, 10(3):347–369, 1982.

[JS03] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, 2. edition, 2003.

[JS04] K. Janeček and S. Shreve. Asymptotic analysis for optimal investment and consumption with transaction costs. *Finance and Stochastics*, 8(2):181–206, 2004.

[Kal00] J. Kallsen. Optimal portfolios for exponential Lévy processes. *Mathematical Methods of Operations Research*, 51(3):357–374, 2000.

[Kal04] J. Kallsen. σ-localization and σ-martingales. *Theory of Probability and Its Applications*, 48(1):152–163, 2004.

[Kal06] J. Kallsen. A didactic note on affine stochastic volatility models. In Y. Kabanov, R. Lipster, and J. Stoyanov, editors, *From Stochastic Calculus to Mathematical Finance*, pages 343–368. Springer, 2006.

[KMK10] J. Kallsen and J. Muhle-Karbe. Exponentially affine martingales, affine measure changes and exponential moments of affine processes. *Stochastic Processes and their Applications*, 120(2):163–181, 2010.

[KMK11] J. Kallsen and J. Muhle-Karbe. Method of moment estimation in time-changed Lévy models. *Statistics and Decisions*, 28(2):169–194, 2011.

[Kor98] R. Korn. Portfolio optimisation with strictly positive transaction costs and impulse control. *Finance and Stochastics*, 2(2):85–114, 1998.

[KP10] J. Kallsen and A. Pauwels. Variance-optimal hedging in general affine stochastic volatility models. *Advances in Applied Probability*, 42(1):83–105, 2010.

[KS02a] J. Kallsen and A. Shiryaev. The cumulant process and Esscher's change of measure. *Finance and Stochastics*, 6(4):397–428, 2002.

[KS02b] J. Kallsen and A. Shiryaev. Time change representation of stochastic integrals. *Theory of Probability and Its Applications*, 46(3):522–528, 2002.

[KS06] D. Kramkov and M. Sîrbu. Sensitivity analysis of utility-based prices and risk-tolerance wealth processes. *The Annals of Applied Probability*, 16(4):2140–2194, 2006.

[KS07] D. Kramkov and M. Sîrbu. Asymptotic analysis of utility-based hedging strategies for small number of contingent claims. *Stochastic Processes and their Applications*, 117(11):1606–1620, 2007.

[KV09] J. Kallsen and R. Vierthauer. Quadratic hedging in affine stochastic volatility models. *Review of Derivatives Research*, 12(1):3–27, 2009.

[KY05] R. Khasminskii and G. Yin. Uniform asymptotic expansions for pricing European options. *Applied Mathematics & Optimization*, 52(3):279–296, 2005.

[Lee04] R. Lee. Option pricing by transform methods: extensions, unification and error control. *The Journal of Computational Finance*, 7(3):51–86, 2004.

[Lew00] A. Lewis. *Option valuation under stochastic volatility*. Finance Press, 2000.

[Lew01] A. Lewis. A simple option formula for general jump-diffusion and other exponential Lévy processes. Envision Financial Systems and OptionCity.net, 2001.

[LK10] R. Lord and C. Kahl. Complex logarithms in Heston-like models. *Mathematical Finance*, 20(4):671–694, 2010.

[MCC98] D. Madan, P. Carr, and E. Chang. The variance gamma process and option pricing. *European Finance Review*, 2(1):79–105, 1998.

[Mer76] R. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1):125–144, 1976.

[MP95] A. Morton and S. Pliska. Optimal portfolio management with fixed transaction costs. *Mathematical Finance*, 5(4):337–356, 1995.

[MS90] D. Madan and E. Seneta. The variance gamma (VG) model for share market returns. *The Journal of Business*, 63:511–524, 1990.

[MS05] M. Mania and M. Schweizer. Dynamic exponential utility indifference valuation. *The Annals of Applied Probability*, 15(3):2113–2143, 2005.

[Pha00] H. Pham. On quadratic hedging in continuous time. *Mathematical Methods of Operations Research*, 51(2):315–339, 2000.

[PPR13] S. Pagliarani, A. Pascucci, and C. Riga. Adjoint expansions in local Lévy models. *SIAM Journal on Financial Mathematics*, 2013. To appear.

[Rai00] S. Raible. *Lévy Processes in Finance: Theory, Numerics and Empirical Facts*. PhD thesis, Universität Freiburg i. Br., 2000.

[Reb99] R. Rebonato. *Volatility and Correlation in the Pricing of Equity, FX, and Interest-rate Options*. John Wiley, 1999.

[Rud64] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 3. edition, 1964.

[Rud87] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 3. edition, 1987.

[RY91] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer, 1991.

[Ryd97] T. Rydberg. The normal inverse Gaussian Lévy process: simulation and approximation. *Communications in Statistics. Stochastic Models*, 13(4):887–910, 1997.

[Sat99] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.

[Sch91] M. Schweizer. Option hedging for semimartingales. *Stochastic Processes and their Applications*, 37(2):339–363, 1991.

[Sch94] M. Schweizer. Approximating random variables by stochastic integrals. *The Annals of Applied Probability*, 22(3):1536–1575, 1994.

[Sch01] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, Handbooks in Mathematical Finance, pages 538–574. Cambridge University Press, 2001.

[Sch03] W. Schoutens. *Lévy Processes in Finance*. John Wiley, 2003.

[SS91] E. Stein and J. Stein. Stock price distributions with stochastic volatility: an analytic approach. *Review of Financial Studies*, 4(4):727–752, 1991.

[SS94] S. Shreve and H. Soner. Optimal investment and consumption with transaction costs. *The Annals of Applied Probability*, 4(3):609–692, 1994.

[SZ99] R. Schöbel and J. Zhu. Stochastic volatility with an Ornstein-Uhlenbeck process: an extension. *European Finance Review*, 3(1):23–46, 1999.

[Tem03] E. Temam. Analysis of error with Malliavin calculus: application to hedging. *Mathematical Finance*, 13(1):201–214, 2003.

[Tof96] K. Toft. On the mean-variance tradeoff in option replication with transactions costs. *Journal of Financial and Quantitative Analysis*, 31(2):233–263, 1996.

[TV09a] P. Tankov and E. Voltchkova. Asymptotic analysis of hedging errors in models with jumps. *Stochastic Processes and their Applications*, 119(6):2004–2027, 2009.

[TV09b] P. Tankov and E. Voltchkova. Jump-diffusion models: a practitioner's guide. *Banque et Marchés*, 99, 2009.

[WDAN05] M. Widdicks, P. Duck, A. Andricopoulos, and D. Newton. The Black-Scholes equation revisited: asymptotic expansions and singular perturbations. *Mathematical Finance*, 15(2):373–391, 2005.

[WW97] A. Whalley and P. Wilmott. An asymptotic analysis of an optimal hedging model for option pricing with transaction costs. *Mathematical Finance*, 7(3):307–324, 1997.

[Zha99] R. Zhang. *Couverture approchée des options Européennes*. PhD thesis, École Nationale des Ponts et Chaussées, 1999.

Acknowledgements

First and foremost, I sincerely thank my advisor Prof. Dr. Jan Kallsen for his guidance and advice on research, teaching, and beyond.

Moreover, I am deeply indebted to Prof. Aleš Černý, PhD, for his invaluable advice on the research about approximate hedging in Lévy models presented in Chapter 4.

I thank Prof. Dr. Johannes Muhle-Karbe for constant support, various valuable comments, and the possibility to present my work in the Mathematical Finance research seminar at ETH Zurich.

Moreover, I sincerely thank my former colleague Dr. Paul Krühner for numerous mathematical discussions. I am indebted to Dr. Sören Christensen and Dr. Stiene Riemer for proofreading and comments.

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertation – abgesehen von der Beratung durch meinen Betreuer Herrn Prof. Dr. Jan Kallsen – nach Inhalt und Form eigenständig angefertigt habe. Dabei habe ich die Regeln guter wissenschaftlicher Praxis der *Deutschen Forschungsgemeinschaft* eingehalten. Die Arbeit hat weder ganz noch zum Teil einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen und ist weder ganz noch zum Teil veröffentlicht oder zur Veröffentlichung eingereicht worden.

Kiel, den 8. März 2013